Resurgence in a Hamilton-Jacobi Equation

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Abstract

We study the resurgent structure associated with a Hamilton-Jacobi equation. This equation is obtained as the inner equation when studying the separatrix splitting problem for a perturbed pendulum via complex matching. We derive the Bridge equation, which encompasses infinitely many resurgent relations satisfied by the formal solution and the other components of the formal integral.

Contents

1 Introduction

1.1 The resurgence phenomenon

1.1.1 The present article is a contribution to Resurgence theory. It is devoted to the first-order partial differential equation

$$\partial_\tau \phi - \frac{1}{8} z^2 (\partial_z \phi)^2 + 2z^{-2}(1 - \mu \sin \tau) = 0 \tag{1}$$

where ϕ is the unknown, the variables are $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ and $z \in \mathbb{C}$, and μ is a complex parameter. This equation must be viewed as the Hamilton-Jacobi equation

$$\partial_\tau \phi + \mathcal{H}(z, \tau, \partial_z \phi) = 0$$

associated with the Hamiltonian function $\mathcal{H}(z, \tau, p) = -\frac{1}{8} z^2 p^2 + 2z^{-2}(1 - \mu \sin \tau)$.

1.1.2 *Resurgence* here means that there exists a divergent formal solution

$$\tilde{\phi}_0(z, \tau) = \sum_{n \geq 0} C_n(\tau) z^{-n-1} \tag{2}$$

whose coefficients are 2π-periodic in τ (they are in fact trigonometric polynomials; of course they depend on μ), and whose formal Borel transform

$$\hat{\phi}_0(\zeta, \tau) = \sum_{n \geq 0} C_n(\tau) \frac{\zeta^n}{n!} \tag{3}$$

converges near the origin and defines a holomorphic function of ζ with analytic continuation along any path of $\mathbb{C}$ which starts from the origin and avoids $i\mathbb{Z}$. The divergence of $\tilde{\phi}_0$ is analyzed through the *alien derivations* Δ_ω; these are operators which measure the singularities of the determinations of $\hat{\phi}_0$ at points ω of $i\mathbb{Z}^*$.

The coefficients $C_n(\tau)$ of $\tilde{\phi}_0$ can be computed inductively, but this is not the case for the alien derivatives $\Delta_\omega \tilde{\phi}_0$ which encode the singularities of $\hat{\phi}_0$ with respect to ζ. However these formal series turn out to be proportional to elementary series:

$$\Delta_\omega \tilde{\phi}_0 = f_0^{[\omega]} \, e^{\omega\tau} \, e^{-\omega \tilde{S}(z,\tau)}, \qquad \omega \in i\mathbb{Z}^*, \tag{4}$$

where the coefficients of $\tilde{S}(z, \tau) = \sum_{n \geq 0} S_n(\tau) z^{-n-1}$ can be computed by induction and only the scalar factors $f_0^{[\omega]}$ remain somewhat mysterious. The divergent series $\tilde{S}$ is very akin to $\tilde{\phi}_0$: its formal Borel transform converges and has the same property of analytic continuation as $\hat{\phi}_0$, and in fact $\tilde{S}$ also stems directly from Eq. (1).

1.1.3 Indeed, looking for a more general object satisfying formally (1), we shall find an array of formal series $\tilde{\phi}_n(z, \tau)$ of the same kind as $\tilde{\phi}_0$, which together give rise to the *formal integral*

$$\tilde{\phi}(z, \tau, c) = \sum_{n \geq 0} c^n \tilde{\phi}_n(z, \tau), \qquad \tilde{\phi}_1(z, \tau) = z - \tau + \tilde{S}(z, \tau), \tag{5}$$

i.e. a double series (in c and z^{-1}, with coefficients which depend periodically on τ except for $\tilde{\phi}_1$) formal solution of (1). The components $\tilde{\phi}_n$ of the formal integral are resurgent functions in z which satisfy *resurgence relations* similar to (4) and involving new coefficients $f_n^{[\omega]}$; this gives access to the singularities of the analytic continuations of their Borel transforms, since the classical rules of alien calculus allow one to compute all the successive alien derivatives $\Delta_{\omega_r}\Delta_{\omega_{r-1}}\ldots\Delta_{\omega_1}\tilde{\phi}_n$ in terms of the "mysterious" coefficients $f_n^{[\omega]}$ and of the series $\tilde{\phi}_n$ themselves.

In other words, we shall obtain a closed system of resurgence relations. Using the generating series $f^{[\omega]}(c) = \sum_{n\geq 0} c^n f_n^{[\omega]}$, they can be grouped into a single equation

$$e^{-\omega z}\Delta_\omega\tilde{\phi}(z,\tau,c) = f^{[\omega]}(c)\,\exp\bigl(-\omega\partial_c\,\tilde{\phi}(z,\tau,c)\bigr). \tag{6}$$

This is the so-called *Bridge equation*, which throws a bridge between alien calculus (the alien derivatives in the left-hand side) and usual differential calculus (here the partial derivative with respect to c in the right-hand side) in the case of the formal integral of an equation.

1.1.4 This phenomenon is typical of Resurgence theory. In fact, the property that the germs $\hat{\phi}_n(\zeta,\tau)$ reappear in such a clear way in their own singular behavior at the points of $i\mathbb{Z}^*$ is the very origin of the name "resurgence", chosen by J. Écalle when he developed his theory [Eca81]. We also refer to [Eca92a, Eca93] or [CNP93a, CNP93b] for an introduction to resurgent functions and alien calculus.

All kinds of local analytic objects (germs of holomorphic vector fields or of holomorphic diffeomorphisms, differential equations, difference equations) fall in the scope of Resurgence. For a pedagogical description of a resurgent formal integral and the Bridge equation it satisfies in some examples, besides the previous references, let us mention [BSSV98] (case of singular Ricatti equations) and [GS01] (detailed study of a second-order difference equation).

To our knowledge, the present study is the first resurgent treatment of a nonlinear partial differential equation.[1] In the framework of ordinary differential equations, "formal integral" means a formal solution which depends on the appropriate number of free parameters; there is an analogous notion for difference equations. Here we use the formal counterpart of what is called a *complete solution* for a first-order partial differential equation (see Sec. 3.1.1).

1.2 Equation (1) as "inner equation"

We present here the Hamiltonian problem which motivates our study of Eq. (1).

1.2.1 Some integrable Hamiltonian systems, when perturbed by a rapidly oscillating term, behave like nearly integrable systems even though the perturbative term is not small: as the frequency of the perturbation becomes larger, the corresponding chaotic zones become extremely small.

Chaos is often measured by the splitting of the separatrices related to a hyperbolic periodic orbit: the angle between those separatrices gives an idea of the magnitude of the chaotic zones appearing near the unperturbed separatrix.

[1]Our problem is very close to the one of [Sau95], but here we adopt the viewpoint of *equational* resurgence which allows us to go much farther (but on a simpler equation) and shall not refer to any *parametric* resurgence.

A typical example which presents this kind of phenomenology is the pendulum with periodic forcing:

$$H_{\mu,\varepsilon}(q,p,t) = \frac{p^2}{2} - 1 + \cos q + \mu(\cos q - 1)\sin(t/\varepsilon) \tag{7}$$

where $\varepsilon > 0$ and μ are two parameters of which the first is assumed to be small but not necessarily the second. The unstable equilibrium of the pendulum gives rise to a hyperbolic $2\pi\varepsilon$-periodic solution whose stable and unstable manifolds do not coincide any longer as was the case with the separatrix of the pendulum.

1.2.2 This system was already considered by Poincaré [Poin93], who wrote the two-dimensional stable and unstable manifolds as graphs of differentials; indeed, being Lagrangian and close to the unperturbed separatrix if $|\mu|$ is small, they admit equations $p = \partial_q S^{\pm}(q,t)$, where S^+ and S^- are $2\pi\varepsilon$-periodic in t and analytic for $q > 0$ small enough in the case of S^-, for $q < 2\pi$ large enough in the case of S^+, and satisfy the Hamilton-Jacobi equation

$$\partial_t S + H_{\mu,\varepsilon}(q, \partial_q S, t) = 0 \tag{8}$$

with asymptotic conditions

$$\lim_{q\to 0} \partial_q S^-(q,t,\mu,\varepsilon) = 0 \text{ (unstable) and } \lim_{q\to 2\pi} \partial_q S^+(q,t,\mu,\varepsilon) = 0 \text{ (stable)} \tag{9}$$

(see for example [Sau95], or [LMS02]). In order to study the possible splitting between the manifolds, we thus need to study the difference between these two particular solutions of the Hamilton-Jacobi equation (8).

In [OS99] (see also [OS00]), it is explained how to apply complex matching methods in this problem. Using the variables $\tau = t/\varepsilon$ and $u = \log\tan(q/4)$, one is led to define the so-called *inner region* by the condition that u be close to $i\pi/2$ (the singularity which is the nearest to the real axis for the time-parameterization $q = 4\arctan e^u$ of the unperturbed separatrix), and to use there the inner variable $z = \frac{u - i\pi/2}{\varepsilon}$. Having performed the change of variables $(q,t) \mapsto (z,\tau)$ in (8), it is then possible to isolate in the resulting equation a dominant part which does not involve the parameter ε: this truncated equation is the so-called inner equation; this is our Eq. (1), which can be viewed as the Hamilton-Jacobi equation associated with the truncated Hamiltonian $\mathcal{H}$.

Eq. (1) admits two particular solutions ϕ^+ and ϕ^- which satisfy for $\Im\mathrm{m}\, z < 0$ asymptotic conditions which parallel (9):

$$\lim_{\Re\mathrm{e}\, z\to\pm\infty} \phi^{\pm}(z,\tau) = 0. \tag{10}$$

It turns out that ϕ^+ and ϕ^- provide sufficiently good approximations of S^+ and S^-, and that an asymptotic estimation of $\phi^+ - \phi^-$ for $|z| \to \infty$ allows one to recover an asymptotic formula for the original separatrix splitting problem as $\varepsilon \to 0$. See [OS99] for more on this.

1.2.3 We shall see in Sec. 2.2 that $\phi^{\pm}$ can be obtained from the formal solution $\tilde\phi_0$ by *Borel-Laplace summation*:

$$\phi^{\pm}(z,\tau) = \int_0^{\pm\infty} e^{-z\zeta}\hat\phi_0(\zeta,\tau)\,d\zeta,$$

and an evaluation of $\phi^+ - \phi^-$ will follow from the first resurgence relation (4). Since, in the case of the first singularity $\omega = i$, (4) can be rephrased as

$$\hat{\phi}_0(i+\xi,\tau) = f_0^{[i]}\, e^{i\tau}\left(\frac{1}{2\pi i\xi} + \hat{\chi}(\xi,\tau)\frac{\log\xi}{2\pi i}\right) + \text{ regular}, \tag{11}$$

where $\hat{\chi}$ is the formal Borel transform of $\tilde{\chi} = -1 + e^{-i\tilde{S}}$, we shall end up with the exponentially small asymptotic equivalent

$$\phi^+ - \phi^- \sim e^{-iz} f_0^{[i]}\, e^{i\tau}(1+\tilde{\chi}(z,\tau)) = f_0^{[i]}\, e^{-i\left(z-\tau+\tilde{S}(z,\tau)\right)}.$$

1.3 Structure of the article

Equation (1) is a particular case of

$$\partial_\tau\phi - \frac{1}{8}z^2(\partial_z\phi)^2 + z^{-2}P_\mu(\tau) = 0,$$

and the study done in this paper can be generalized to another equation of this family. The proofs will be particularly similar when $P_\mu(\tau)$ has some symmetric properties (see, for instance, Lemma 2), but we think that this study can deal with nonsymmetric perturbations by allowing $\hat{\phi}_0$ to have more complicated singularities.

The treatment of other Hamilton-Jacobi equations coming from a rapidly perturbed forced pendulum, will need a more detailed study in order to obtain a inner equation susceptible to be examined in the scope of Resurgent Theory.

This paper is organized as follows. In Sec. 2.1 we prove the existence of a formal solution $\tilde{\phi}_0$ of Eq. (1) and we consider its Borel transform $\hat{\phi}_0$. Theorem 1 establishes the properties of the analytic continuation of $\hat{\phi}_0$ in the main sheet of its Riemann surface, as well as in the nearby sheets. The shape of $\hat{\phi}_0$ near its first singularity i is stated in Theorem 2. The proofs of both theorems are deferred to Sections 2.3 and 2.4 respectively.

In Sec. 2.2 we apply Theorem 1 and 2 to obtain Cor. 1 about the existence of functions $\phi^\pm$ which solve equation (1), and the value of their difference $\phi^+ - \phi^-$.

In order to give the complete resurgent structure of $\tilde{\phi}_0$, in Sec. 3.1 we present the so-called formal integral and formulate Theorem 3, which asserts that all the components of the formal integral verify some resurgent relations. All these relations can be expressed in a compact way writing the Bridge equation, which is the main tool to prove that they are resurgent functions. In Sec. 3.2 we prove some technical results which are used in Sec. 3.3 devoted to the proof of Theorem 3.

At the occasion of this workshop in his honor, it is a pleasure to acknowledge Frédéric Pham's influence and the help that his work on Resurgence offers to whom wishes to use this theory.

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2 Study of the Borel transform $\hat{\phi}_0$ of the formal solution

2.1 Statement of the results

2.1.1 Formal solutions of Eq. (1)

We shall work with formal series in z^{-1} whose coefficients are trigonometric polynomials of τ, i.e. elements of $\mathcal{P}[[z^{-1}]]$, where $\mathcal{P}$ stands for the space $\mathbb{C}[[e^{i\tau}, e^{-i\tau}]]$ of trigonometric polynomials. All our formal series will also depend on the complex parameter μ; in fact their coefficients will be entire functions of μ.

By $z^{-1}\mathcal{P}[[z^{-1}]]$, we shall denote the space of expansions in negative powers of z, i.e. formal series of which the z-independent term vanishes, and the largest space of formal series we shall consider in Sec. 3 will be $\mathcal{P}[z][[z^{-1}]]$ (sums of polynomials in z and formal series in z^{-1} all of whose coefficients depend on τ as trigonometric polynomials).

Thus all the formal series to be found in the sequel may be expanded in two ways:

$$\tilde{\varphi}(z,\tau) = \sum_{n\geq n_0} \varphi_n(\tau) z^{-n} = \sum_{k\in\mathbb{Z}} \tilde{\varphi}^{[k]}(z) e^{ik\tau},$$

where n_0 is some integer (positive in the case of $z^{-1}\mathcal{P}[[z^{-1}]]$, possibly negative in the case of $\mathcal{P}[z][[z^{-1}]]$), and the φ_n's belong to $\mathcal{P}$ whereas each $\tilde{\varphi}^{[k]}$ can be written

$$\tilde{\varphi}^{[k]}(z) = \sum_{n\geq n_0} \varphi_n^{[k]} z^{-n},$$

with scalar coefficients $\varphi_n^{[k]}$. As usual, we define the formal Borel transform $\mathcal{B}$ as the linear operator of $z^{-1}\mathbb{C}[[z^{-1}]]$ or of $z^{-1}\mathcal{P}[[z^{-1}]]$ whose action on such formal series reads

$$\mathcal{B}(\tilde{\varphi}^{[k]})(\zeta) = \hat{\varphi}^{[k]}(\zeta) = \sum_{n\geq n_0} \varphi_n^{[k]} \frac{\zeta^{n-1}}{(n-1)!} \quad \text{or} \quad \mathcal{B}(\tilde{\varphi})(\zeta,\tau) = \hat{\varphi}(\zeta,\tau) = \sum_{n\geq n_0} \varphi_n(\tau) \frac{\zeta^{n-1}}{(n-1)!}$$

(where $n_0 \geq 1$; when necessary, this action is extended to $\mathbb{C}[z][[z^{-1}]]$ or $\mathcal{P}[z][[z^{-1}]]$ by use of the Dirac mass at 0 and its derivatives: $z^j \mapsto \delta^{(j)}$ if $j \geq 0$—see [Eca81], [CNP93a] and Sec. 2.4.2 below).

Lemma 1 *For each $\mu \in \mathbb{C}$, the solutions in $\mathcal{P}[z][[z^{-1}]]$ of the Hamilton-Jacobi equation (1) are of the form*

$$\phi(z,\tau) = \alpha + \tilde{\phi}_0(z,\tau;\mu) \quad \textit{or} \quad \alpha + \tilde{\phi}_0(-z,\tau;\mu),$$

where α is an arbitrary complex number and

$$\tilde{\phi}_0 = \sum_{n\geq 0} C_n(\tau;\mu) z^{-n-1} \tag{12}$$

is determined as the unique solution in $z^{-1}\mathcal{P}[[z^{-1}]]$ with leading term $4z^{-1}$. Its coeffi-

cients are determined by the recursion formulae $C_0 = 4$, $C_1 = -2\mu\cos\tau$ and

$$< C_n > = -\frac{1}{8(n+1)} \sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 1}} < (n_1+1)C_{n_1}(n_2+1)C_{n_2} >, \tag{13}$$

$$\partial_\tau C_n = \frac{1}{8} \sum_{\substack{n_1+n_2=n-1 \\ n_1,n_2\geq 0}} (n_1+1)C_{n_1}(n_2+1)C_{n_2}, \tag{14}$$

for $n \geq 2$, where $< . >$ denotes the mean value of a trigonometric polynomial.

Observe that the right-hand side in (14) has zero mean value, so that (13) and (14) do define a unique trigonometric polynomial C_n (which depends polynomially on μ). Here are the first few terms

$$\begin{aligned}\tilde{\phi}_0(z,\tau;\mu) &= 4z^{-1} - (2\mu\cos\tau)z^{-2} \\ &\qquad - (4\mu\sin\tau + \frac{1}{3}\mu^2)z^{-3} + (12\mu\cos\tau + \frac{1}{2}\mu^2\sin 2\tau)z^{-4} + O(z^{-5})\end{aligned}$$

The proof of this lemma is left to the reader. Of course the possibility of adding a constant corresponds to the fact that Eq. (1) involves only the partial derivatives of the unknown function. The choice of the sign 'plus' or 'minus' in front of z is more significant: if we think of (1) as an approximative Hamilton-Jacobi equation for the perturbed pendulum, this means that we can study the splitting of the upper part of the unperturbed separatrix as well as of the lower part ($p = \pm 2\sin(q/2)$ with the notations of Sec. 1.2). This corresponds to a symmetry of the equation with respect to the substitution $z \mapsto -z$. We choose the 'plus' sign in all the sequel.

Our equation (1) presents another symmetry which will simplify certain details of the analysis below and which corresponds to the symmetry with respect to the substitution $\tau \mapsto \pi - \tau$ of the perturbation $P_\mu(\tau) = 2(1-\mu\sin\tau)$ we have chosen (cf. Sec 1.3).

Lemma 2 *The formal solution $\tilde{\phi}_0$ of Eq. (1) is antisymmetric with respect to the involution $(z,\tau) \mapsto (-z, \pi-\tau)$:*

$$\tilde{\phi}_0(-z,\pi-\tau;\mu) = -\tilde{\phi}_0(z,\tau;\mu).$$

The proof is immediate, in view of the symmetry of the equation and of the unicity statement in Lemma 1. We observe in the same way that in our case there is also a symmetry with respect to μ:

$$\tilde{\phi}_0(z,\tau;-\mu) = \tilde{\phi}_0(z,\pi+\tau;\mu) \tag{15}$$

(due to the relation $P_\mu(\tau) = P_{-\mu}(\tau-\pi)$).

2.1.2 The Riemann surface $\mathcal{R}$

We shall be interested in the convergence and the analytic continuation of $\hat{\phi}_0 = \mathcal{B}\tilde{\phi}_0$ with respect to the variable ζ. According to the classical properties of the Borel transform, $\mathcal{B}(\tilde{\phi}\,\tilde{\psi}) = \mathcal{B}(\tilde{\phi}) * \mathcal{B}(\tilde{\psi})$ with

$$(\hat{\phi} * \hat{\psi})(\zeta) = \int_0^\zeta \hat{\phi}(\zeta_1)\hat{\psi}(\zeta-\zeta_1)\,d\zeta_1 \tag{16}$$

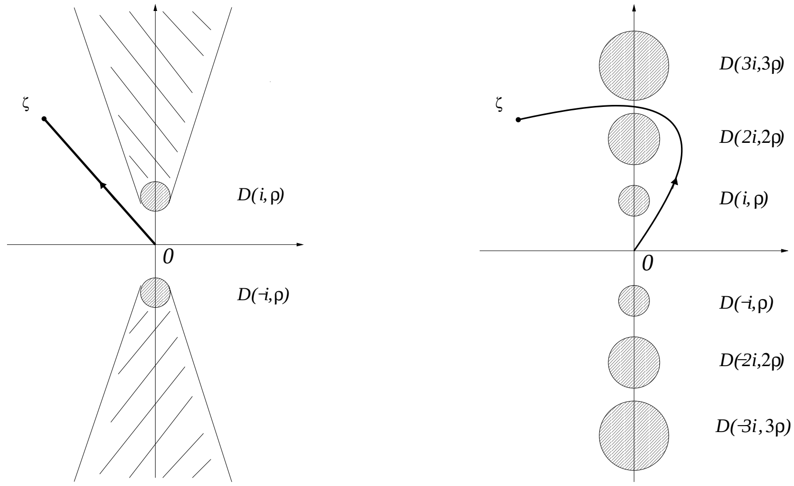

Figure 1: Examples of points of the subsets $\mathcal{R}_\rho^{(0)}$ and $\mathcal{R}_\rho^{(1)}$ of $\mathcal{R}$

and $\mathcal{B}(z\partial_z\tilde{\phi}_0) = -\partial_\zeta(\zeta\hat{\phi}_0)$, hence Eq. (1) transforms into

$$\partial_\tau\hat{\phi}_0 - \frac{1}{8}\left(\hat{\phi}_0 + \zeta\partial_\zeta\hat{\phi}_0\right)^{*2} + 2\zeta(1-\mu\sin\tau) = 0. \tag{17}$$

As indicated in Sec. 1.1.2, we will see that the only possible singularities of the analytic continuation of $\hat{\phi}_0$ will turn out to lie on $i\mathbb{Z}$. It is convenient to express this as the property of being holomorphic on the Riemann surface $\mathcal{R}$ consisting of all homotopy classes[2] of paths issuing from 0 and lying in $\mathbb{C}\setminus i\mathbb{Z}$ (except their origin). We denote by $\zeta\in\mathcal{R}\mapsto\overset{\bullet}{\zeta}\in(\mathbb{C}\setminus i\mathbb{Z})\cup\{0\}$ the natural projection, which is biholomorphic at each point ($\overset{\bullet}{\zeta}$ is the extremity of any path representing ζ; only the origin projects onto 0 and this is the only difference between $\mathcal{R}$ and the universal cover of $\mathbb{C}\setminus i\mathbb{Z}$).

In Sec. 2.1 we shall confine ourselves to subsets of $\mathcal{R}$ only. Analytic continuation of $\hat{\phi}_0$ in the whole of the Riemann surface is deferred to Sec. 3, because it will not be obtained independently of the study of the other components $\tilde{\phi}_n$ of the formal integral mentioned in the introduction.

The *main sheet* $\mathcal{R}^{(0)}$ *of* $\mathcal{R}$ is obtained as the subset of the points ζ of $\mathcal{R}$ which can be represented by the straight segment $[0,\overset{\bullet}{\zeta}]$; it is isomorphic to the cut plane $\mathbb{C}\setminus\big(\pm i\,[1,\infty[\big)$. For $\rho\in\,]0,1[$, we define $\mathcal{R}_\rho^{(0)}$ by "thickening" the singular half-lines $\pm i\,[1,\infty[$ and considering open discs $D(i,\rho)$ and $D(-i,\rho)$ of radius ρ centered at i and $-i$ (see Figure 1):

$$\mathcal{R}_\rho^{(0)} = \{\,\zeta\in\mathcal{R} \text{ represented by } [0,\overset{\bullet}{\zeta}]\subset\mathbb{C}\setminus D(\pm i,\rho)\,\}. \tag{18}$$

By $\mathcal{R}^{(1)}$ we denote the union of $\mathcal{R}^{(0)}$ and of the "nearby half-sheets", i.e. the half-sheets which are contiguous to the main one: this is the subset of $\mathcal{R}$ consisting of the homotopy classes of paths still issuing from 0 and lying in $\mathbb{C}\setminus i\mathbb{Z}$ but crossing at most once the imaginary axis (no crossing at all means we stay in the main sheet, but we

[2]When mentioning homotopy of paths, we always refer to homotopy with fixed extremities; we follow here the notations of [GS01] up to minor details.

arrive to a new half-sheet each time we cross between two consecutive singular points mi and $(m+1)i$, or $-(m+1)i$ and $-mi$, with $m \geq 1$).

Analogously to the auxiliary sets $\mathcal{R}_\rho^{(0)}$ (which will be used to prove analyticity in $\mathcal{R}^{(0)}$, by considering arbitrarily small ρ), we shall define subsets $\mathcal{R}_\rho^{(1)}$ of $\mathcal{R}$ whose union covers $\mathcal{R}^{(1)}$ and which will be used in the study of the analyticity and growth of $\hat{\phi}_0$ in $\mathcal{R}^{(1)}$. However, since their definition is quite technical and is a mere adaptation of [GS01], we delay it to Sec. 2.3.3, although we refer to them in the statement of Theorem 1. See Figure 1: points in $\mathcal{R}_\rho^{(1)}$ can be represented by a path which stays in $\mathcal{R}_\rho^{(0)}$ or which passes between the discs $D(\pm mi, m\rho)$ and $D\big(\pm(m+1)i, (m+1)\rho\big)$ with $1 \leq m < \frac{1}{2}(\rho^{-1}-1)$ and crosses the imaginary axis at most once. Of course this requires $\rho < \frac{1}{3}$.

2.1.3 Statement of the first analyticity result for $\hat{\phi}_0$

It will be convenient to use the auxiliary formal series $\tilde{F}(z,\tau;\mu) = (\cos\tau)z^{-1} + (3\sin\tau + \frac{1}{4}\mu)z^{-2} + O(z^{-3})$ defined through its formal Borel transform by

$$\hat{\phi}_0(\zeta,\tau;\mu) = 4 - 4\mu\zeta^{-1}\big(\zeta * \hat{F}(\zeta,\tau;\mu)\big). \tag{19}$$

Theorem 1 *The formal Borel transforms $\hat{\phi}_0(\zeta,\tau;\mu)$ and $\hat{F}(\zeta,\tau;\mu)$ are convergent for ζ close to the origin (uniformly in τ and μ). The resulting holomorphic functions of the three variables ζ, τ and μ (still denoted by $\hat{\phi}_0$ and $\hat{F}$) admit an analytic continuation in $\mathcal{R}^{(1)} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$. Moreover, for each $\rho \in \left]0, \frac{1}{3}\right[$, there exists a continuous function ℓ on $\mathcal{R}_\rho^{(1)}$ such that*

$$\forall (\zeta,\tau,\mu) \in \mathcal{R}_\rho^{(1)} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}, \quad |\hat{F}(\zeta,\tau;\mu)| \leq 2\rho^{-3}\cosh(\Im\mathrm{m}\,\tau)\, e^{c\,\ell(\zeta)}, \tag{20}$$

where $c = 24\rho^{-2}\max\Big(\rho^{-2}|\mu|\cosh(\Im\mathrm{m}\,\tau), \big(|\mu|\cosh(\Im\mathrm{m}\,\tau)\big)^{1/3}\Big)$ and

$$\big|\dot{\zeta}\big| \leq \ell(\zeta) \leq (2m+1)\big|\dot{\zeta}\big| + 12m(m+1), \tag{21}$$

where the integer m indicates on which sheet of $\mathcal{R}$ the point ζ lies: $m = 0$ if $\zeta \in \mathcal{R}^{(0)}$, and if not, m is determined by the necessity of crossing $[mi, (m+1)i]$ or $[-mi, -(m+1)i]$ to represent ζ (necessarily $1 \leq m < \frac{1}{2}(\rho^{-1}-1)$).

The proof is the subject of Sec. 2.3 below.

Remark 1 We are not particularly interested in complex values of τ, but since we shall work with Fourier expansions this does not make any difference: the results which will be obtained can be specialized to real values of τ at the end.

Remark 2 The auxiliary function $\hat{F}$ satisfies an equation which is easier to study and we shall obtain directly the bounds (20) when proving Theorem 1. It is straightforward to deduce from them analogous estimates for the function $\hat{\phi}_0$: according to Lemma 7 in Sec. 2.2, for $(\zeta,\tau,\mu) \in \mathcal{R}_\rho^{(1)} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$,

$$|\hat{\phi}_0(\zeta,\tau;\mu)| \leq 4 + 4\rho^{-3}\cosh(\Im\mathrm{m}\,\tau)\,\frac{|\mu|\ell(\zeta)^2}{\big|\dot{\zeta}\big|}\, e^{c\,\ell(\zeta)}. \tag{22}$$

2.1.4 Statement relative to the singularity at $\zeta = i$

Let us now focus on the singularity of $\hat{\phi}_0(\zeta, \tau; \mu)$ at $\zeta = i$. Obviously, all what follows could be done for the singularity at $-i$ analogously, or one can use the antisymmetry of $\tilde{\phi}_0$: Lemma 2 implies that $\hat{\phi}_0(-\zeta, \tau; \mu) = \hat{\phi}_0(\zeta, \pi - \tau; \mu)$.

As announced in the introduction (Formulae (4) and (11)), the singular behavior of $\hat{\phi}_0$ will be related to a formal series $\tilde{S}$ whose coefficients can be computed by induction.

Lemma 3 *The linear equation (linearization of Eq. (1) around $\tilde{\phi}_0$)*

$$\partial_\tau Y - \frac{1}{4} z^2 (\partial_z \tilde{\phi}_0) \partial_z Y = 0 \tag{23}$$

admits a unique solution of the form

$$Y = z - \tau + \tilde{S}, \qquad \tilde{S} \in z^{-1}\mathcal{P}[[z^{-1}]]. \tag{24}$$

We have

$$\tilde{S} = \tilde{S}(z, \tau; \mu) = (-\frac{1}{4}\mu^2 + \mu \sin \tau) z^{-1} - 4\mu (\cos \tau) z^{-2} + O(z^{-3}).$$

Remark 3 This formal series $\tilde{S}$ is antisymmetric with respect to the involution $(z, \tau) \mapsto (-z, \pi - \tau)$ because, $z^2 \partial_z \tilde{\phi}_0$ being symmetric, whenever $Y(z, \tau) = z - \tau + \tilde{S}(z, \tau)$ satisfies (23), $-\pi - Y(-z, \pi - \tau) = z - \tau + \tilde{S}(-z, \pi - \tau)$ is also solution.

Proof. According to (19), the series $\tilde{F} = \sum_{n \geq 0} F_n(\tau) z^{-n-1}$ satisfies

$$-\frac{1}{4} z^2 \partial_z \tilde{\phi}_0 = 1 - \mu \tilde{F}, \tag{25}$$

thus the equation for $\tilde{S}$ is $(\partial_\tau + \partial_z)\tilde{S} = \mu \tilde{F} + \mu \tilde{F}\, \partial_z \tilde{S}$. Plugging $\tilde{S} = \sum_{n \geq 0} S_n(\tau) z^{-n-1}$ inside, we find the system of equations

$$\partial_\tau S_0 = \mu F_0 \tag{26}$$

$$\partial_\tau S_n = n S_{n-1} + \mu F_n - \mu \sum_{n_1 + n_2 = n-1} (n_2 + 1) F_{n_1} S_{n_2}, \qquad n \geq 1 \tag{27}$$

(by convention, the empty sum in the right-hand side of (27) at rank $n = 1$ means 0).

There is no obstruction to solve (26) because $F_0 = \cos \tau$ has zero mean value (as should be due to symmetry: the τ-average of a symmetric series is always an even series in z^{-1}). It is then easy to check that this system of equations admits a solution, which is uniquely determined by supplementing (26) and (27) with

$$< S_n >= \frac{\mu}{n+1} < -F_{n+1} + \sum_{n_1 + n_2 = n-2} (n_2 + 1) F_{n_1} S_{n_2} >, \qquad n \geq 0, \tag{28}$$

(indeed, (28) at rank n follows from (27) at rank $n + 1$, but then the right-hand side in (27) at rank n has zero mean value thanks to (28) at rank $n - 1$) and to compute the first few terms. □

We shall see in Sec. 3.1.2 another way of obtaining the series $\tilde{S}$. Of course, its formal Borel transform $\hat{S}$ too will be convergent for ζ close to the origin and will extend analytically to $\mathcal{R}$.

Theorem 2 *There exist functions $A(\tau;\mu)$, $\hat{\psi}(\xi,\tau;\mu)$ and $\hat{r}(\xi,\tau;\mu)$, which are holomorphic for $(\zeta,\tau,\mu)\in\mathcal{R}^{(0)}\times(\mathbb{C}/2\pi\mathbb{Z})\times\mathbb{C}$, such that*

$$\hat{\phi}_0(i+\xi,\tau;\mu)=\frac{A(\tau;\mu)}{2\pi i\xi}+\hat{\psi}(\xi,\tau;\mu)\frac{\log\xi}{2\pi i}+\hat{r}(\xi,\tau;\mu), \tag{29}$$

for $i+\xi$ staying on the main sheet $\mathcal{R}^{(0)}$ (i.e. the above equation describes the singularity at i of the principal determination of $\hat{\phi}_0$).

Moreover, there exists an odd entire function $f_0^{[i]}(\mu)=-2\pi i\mu+O(\mu^3)$ such that

$$A(\tau;\mu)+\tilde{\psi}(z,\tau;\mu)=f_0^{[i]}(\mu)\,e^{i\tau}e^{-i\tilde{S}(z,\tau;\mu)} \tag{30}$$

where $\tilde{\psi}=\mathcal{B}^{-1}\hat{\psi}$.

The proof is given in Sec. 2.4 below.

By definition[3] of the alien derivation Δ_i (and because i is the first singular point we meet when we move along $i\mathbb{R}^+$ starting from 0), Eq. (29) is equivalent to

$$\Delta_i\tilde{\phi}_0=A+\tilde{\psi}. \tag{31}$$

In the case of $\omega=i$, the relation (4) announced in the introduction can thus be considered as a resurgent formulation of Eqns. (29) and (30). Observe that $A(\tau;\mu)$, which can be defined as the residuum of the polar part of $\hat{\phi}_0$ at $\zeta=i$ (up to the factor $2\pi i$) and to some extent computed as such, reduces to

$$A(\tau;\mu)=f_0^{[i]}(\mu)\,e^{i\tau}.$$

The oddness of $f_0^{[i]}$ is a special feature of our problem which follows from (15).

2.2 Borel-Laplace sums of $\tilde{\phi}_0$ and application to the separatrix splitting problem

Theorems 1 and 2 are sufficient to implement the method alluded to in Sec. 1.2.3 to study particular analytic solutions of the Hamilton-Jacobi equation (1).

When restricting to $\mathcal{R}_\rho^{(0)}\times(\mathbb{C}/2\pi\mathbb{Z})\times\mathbb{C}$ and identifying $\mathcal{R}_\rho^{(0)}$ with a subset of $\mathbb{C}$, we must use $\ell(\zeta)=|\zeta|$ in the bound (22); this bound thus indicates that $\hat{\phi}_0(\zeta,\tau;\mu)$ has at most exponential growth with respect to ζ, with exponential type not larger than

$$c=c_\rho(\tau;\mu)=24\rho^{-2}\max\left(\rho^{-2}|\mu|\cosh(\Im\mathrm{m}\,\tau),\left(|\mu|\cosh(\Im\mathrm{m}\,\tau)\right)^{1/3}\right).$$

Thus, for any $\theta\in\left]-\pi/2,3\pi/2\right]$ such that the ray $\left[0,e^{i\theta}\infty\right[$ is contained in $\mathcal{R}_\rho^{(0)}$, i.e. such that

$$\theta\in\mathcal{I}_\rho^+=\left[-\tfrac{\pi}{2}+\arcsin\rho,\tfrac{\pi}{2}-\arcsin\rho\right]\quad\text{or}\quad\theta\in\mathcal{I}_\rho^-=\left[\tfrac{\pi}{2}+\arcsin\rho,\tfrac{3\pi}{2}-\arcsin\rho\right],$$

[3] In fact we use here a slight generalization of the classical operator Δ_i introduced by Écalle: when a formal series $\tilde{\phi}$ is known to belong to the algebra RES of resurgent functions, in the sense of having a Borel transform holomorphic in $\mathcal{R}$, the alien derivatives $\Delta_\omega\tilde{\phi}$ are defined for all nonzero complex numbers ω (only integer values of ω may yield a nonzero result) and are themselves resurgent functions. Here, the formal solution $\tilde{\phi}_0$ is not yet known to belong to RES, but we can define, like in [GS01, Sec. 5.4], a larger space $\mathrm{RES}^{(1)}$ which contains it and on which Δ_i acts as a derivation (but Δ_i sends $\mathrm{RES}^{(1)}$ in a still larger space SING)—see Sec. 2.4.2 for more on this. This is in fact a simple extrapolation from [Eca92b] (beginning of Sec. 2.1).

we can consider the Laplace integral

$$\mathcal{L}^\theta \hat\phi_0(z,\tau,\mu) = \int_0^{e^{i\theta}\infty} e^{-z\zeta}\hat\phi_0(\zeta,\tau;\mu)\,d\zeta$$

provided $\Re\mathrm{e}(z\,e^{i\theta}) > c = c_\rho(\tau;\mu)$. By virtue of the Cauchy theorem, this defines two holomorphic functions ϕ^+ and ϕ^-. We shall denote by $\phi^\pm$ the one which is obtained by gluing the functions $\mathcal{L}^\theta\hat\phi_0$ with $\theta \in \mathcal{I}_\rho^\pm$; it is defined and holomorphic in $\{\,(z,\tau;\mu) \in \mathbb{C}\times(\mathbb{C}/2\pi\mathbb{Z})\times\mathbb{C} \mid z \in \mathcal{D}_\rho^\pm(\tau;\mu)\,\}$ for every $\rho \in \,]0,1[$, where $\mathcal{D}_\rho^\pm(\tau;\mu)$ is obtained as the union of the domains $\Re\mathrm{e}(z\,e^{i\theta}) > c$ for $\theta \in \mathcal{I}_\rho^\pm$, that is

$$\mathcal{D}_\rho^\pm(\tau;\mu) = c_\rho(\tau;\mu)\Sigma_\rho^\pm, \qquad \Sigma_\rho^\pm = \{\,z\in\mathbb{C} \mid \forall z' \in [\pm\rho^{-1}, z],\ |z'| \geq 1\,\}$$

(see Figure 2). For technical reasons, we shall also use the smaller domains

$$\underline{\mathcal{D}}_\rho^\pm(\tau;\mu) = \big(c_\rho(\tau;\mu)+1\big)\Sigma_{2\rho}^\pm, \quad \text{and} \quad \underline{\mathcal{D}}_\rho(\tau;\mu) = \big(3c_\rho(\tau;\mu)+1\big)\big(\Sigma_{2\rho}^+\cap\Sigma_{2\rho}^-\big).$$

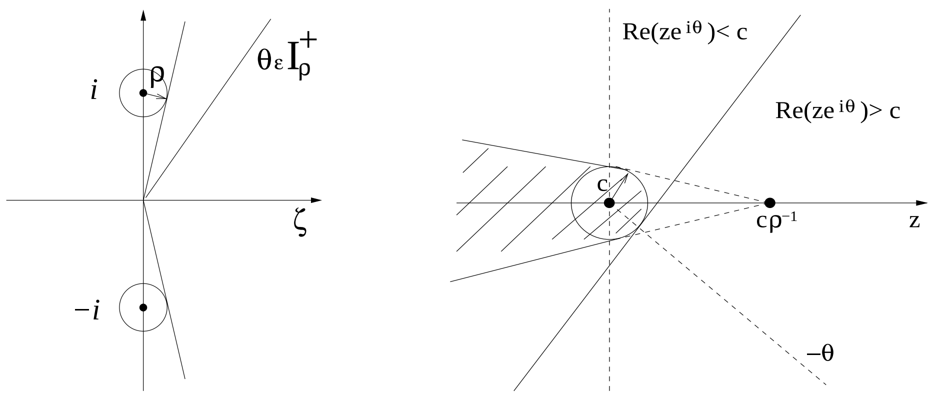

Figure 2: The condition $z \in \mathcal{D}_\rho^+ = c\,\Sigma_\rho^+$ is imposed when performing the Laplace transforms in directions $\theta \in \mathcal{I}_\rho^+$

Corollary 1 *The functions $\phi^\pm(z,\tau;\mu)$ just defined are solutions of Eq. (1). They satisfy the condition (10), and in fact they admit the formal solution $\tilde\phi_0$ of Lemma 1 as asymptotic expansion: for each $\rho \in \left]0,\frac{1}{3}\right[$, $\mu_0 > 0$, $\tau_0 \geq 0$,*

$$\phi^\pm(z,\tau;\mu) \sim \sum_{n\geq 0} C_n(\tau;\mu) z^{-n-1} \quad \text{as } z \in \underline{\mathcal{D}}_\rho^\pm(i\tau_0;\mu_0), \tag{32}$$

uniformly for $|\mu| \leq \mu_0$ and $|\Im\mathrm{m}\,\tau| \leq \tau_0$.

Their difference is exponentially small as $\Im\mathrm{m}\,z \to \pm\infty$: with the notations of Theorem 2,

$$\phi^+(z,\tau;\mu) - \phi^-(z,\tau;\mu) \sim f_0^{[i]}(\mu)\,e^{-i\left(z-\tau+\tilde S(z,\tau;\mu)\right)} \tag{33}$$

as $z \in \underline{\mathcal{D}}_\rho(i\tau_0;\mu_0)\cap\{\Im\mathrm{m}\,z < 0\}$, again uniformly for $|\mu| \leq \mu_0$ and $|\Im\mathrm{m}\,\tau| \leq \tau_0$.

In particular, we have an asymptotic equivalent for $\Im\mathrm{m}\,z$ tending to $-\infty$ and μ tending to 0 independently, uniformly for $|\Im\mathrm{m}\,\tau| \leq \tau_0$,

$$\phi^+(z,\tau;\mu) - \phi^-(z,\tau;\mu) \sim -2\pi i\mu e^{-i(z-\tau)}.$$

Remark 4 Lemma 2 implies that $\phi^-(z,\tau;\mu) = -\phi^+(-z,\pi-\tau;\mu)$. The splitting problem thus amounts to measuring the defect of antisymmetry of ϕ^+ or ϕ^-, while their common asymptotic expansion is antisymmetric.

Remark 5 In fact, more than in the solutions ϕ^+ and ϕ^- themselves, we are interested in their partial derivatives: it is $\partial_z\phi^\pm$ which corresponds to $\partial_q S^\pm$ and thus to the geometric object (stable or unstable manifold); intersections of the manifolds take place at critical points of $S^+ - S^-$ and the corresponding angle of splitting is measured by $\partial_q^2(S^+ - S^-)$. But it will be clear that the asymptotic Formula (33) can be differentiated with respect to z.

Proof. The functions ϕ^+ and ϕ^- were obtained from the solution $\tilde\phi_0$ of (1) by Borel-Laplace summation, whose general properties ensure that they admit $\tilde\phi_0$ as asymptotic expansion and satisfy the same equation. As for the uniformity statement in (32), what we mean is that, for each $n \geq 0$, the function

$$z^{n+1}\big(\phi^\pm(z,\tau;\mu) - \sum_{j=0}^{n-1} C_j(\tau;\mu) z^{-j-1}\big)$$

is bounded in $\{\,(z,\tau;\mu) \mid z \in \underline{\mathcal{D}}^\pm_\rho(i\tau_0;\mu_0),\ |\Im\mathrm{m}\,\tau| \leq \tau_0\ |\mu| \leq \mu_0\,\}$ (in fact, this is a Gevrey-1 asymptotic expansion: this function is bounded by $\mathrm{const}\,\rho^{-n}(n+1)!$). It is sufficient to follow here [Mal95, Sec. 1.4.2], treating τ and μ as parameters with respect to which uniformity is obtained by fixing ρ, μ_0, τ_0. Indeed, the inequality (22) yields

$$\forall \zeta \in \mathcal{R}^{(0)}_\rho, \quad |\hat\phi_0(\zeta,\tau;\mu)| \leq (A + B|\zeta|)e^{C|\zeta|} \tag{34}$$

with $A, B, C > 0$ depending only on ρ, μ_0, τ_0, provided $|\mu| \leq \mu_0$ and $|\Im\mathrm{m}\,\tau| \leq \tau_0$. In particular, one can take $C = c_\rho(i\tau_0,\mu_0)$, and if $z \in \underline{\mathcal{D}}^+_\rho(i\tau_0,\mu_0)$ one can find $\theta \in \mathcal{I}^+_{2\rho}$ such that $\Re\mathrm{e}(z\,e^{i\theta}) > C+1$, hence the inequality (34) is satisfied in the strip $\{\mathrm{dist}(\zeta, e^{i\theta}\mathbb{R}^+) \leq \rho\}$ and the asymptotic estimates for ϕ^+ follow like in [Mal95]. The case of ϕ^- is analogous.

We now move on to the proof of (33), which is a uniform asymptotic expansion of the same kind as previously except for the exponentially small factor e^{-iz}. We fix $\rho < \frac{1}{3}$, $a \in\,]1+\rho, 2-2\rho[$, and μ_0, τ_0; we still denote by C the number $c_\rho(i\tau_0;\mu_0)$. Let $(\tau;\mu) \in (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$ such that $|\Im\mathrm{m}\,\tau| \leq \tau_0$ and $|\mu| \leq \mu_0$: for $\theta \in \big]0, \frac{\pi}{2} - \arcsin(2\rho)\big[$, we consider the difference

$$\phi^+(z,\tau;\mu) - \phi^-(z,\tau;\mu) = \int_{e^{i(\pi-\theta)}\infty}^{e^{i\theta}\infty} e^{-z\zeta}\hat\phi_0(\zeta,\tau;\mu)\,d\zeta \tag{35}$$

for z such that $\Re\mathrm{e}(z\,e^{i\theta}) > 3C+1$, $\Re\mathrm{e}(z\,e^{i(\pi-\theta)}) > 3C+1$ and $\Im\mathrm{m}\,z < 0$ (letting θ vary, we would thus cover $\underline{\mathcal{D}}_\rho \cap \{\Im\mathrm{m}\,z < 0\}$).

Applying the Cauchy Theorem, we can push the path of integration upwards as long as we do not reach i. The analytic continuation of $\hat\phi_0$ in the nearby half-sheets of the Riemann surface $\mathcal{R}$ allows us to deform this path, crossing the imaginary axis between i and $2i$ and even going to infinity forwards and then backwards before returning on the main sheet as shown on Figure 3. Indeed, the possibility of going to infinity in the corresponding half-sheet still preserving the convergence of our integral is guaranteed by (22) with $m = 1$, i.e. $\ell(\zeta) \leq 3|\zeta| + 24$, which yields

$$|\hat\phi_0(\zeta,\tau;\mu)| \leq (A + B|\overset{\bullet}{\zeta}|)e^{3C|\overset{\bullet}{\zeta}|} \tag{36}$$

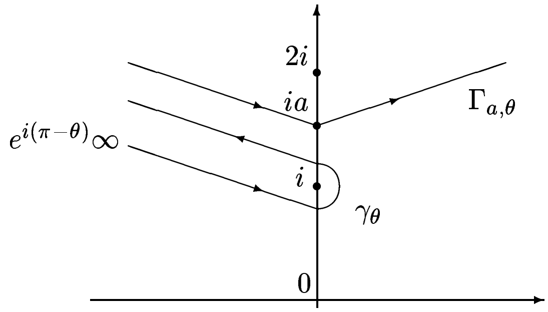

Figure 3: Deformation of contour for the study of $\phi^+ - \phi^-$

for the points ζ of interest, with A, B depending only on ρ, μ_0, τ_0.

So, the integral will decompose into the contribution of the singularity at i (integral on γ_θ) and an exponentially smaller term (integral on $\Gamma_{a,\theta}$):

$$\phi^+(z,\tau;\mu) - \phi^-(z,\tau;\mu) = \int_{\gamma_\theta} \hat{\phi}_0(\zeta,\tau;\mu)\, e^{-z\zeta}\, d\zeta + \int_{\Gamma_{a,\theta}} \hat{\phi}_0(\zeta,\tau;\mu)\, e^{-z\zeta}\, d\zeta.$$

Indeed, using (36) and the condition on $\Re\mathrm{e}(z\, e^{i\theta})$, we see that the second term is bounded by $D\, e^{-a|\Im\mathrm{m}\, z|}$, where D depends only on ρ, τ_0, μ_0 (by factorizing e^{-iaz}).

In view of (29), the integral on γ_θ can be written as

$$e^{-iz} A(\tau;\mu) + e^{-iz} \int_0^{e^{i(\pi-\theta)}\infty} \hat{\psi}(\xi,\tau;\mu)\, e^{-z\xi}\, d\xi$$

and is thus asymptotic to $e^{-iz}\big(A(\tau;\mu) + \tilde{\psi}(z,\tau;\mu)\big)$, which—according to (30)—coincides with $f_0^{[i]}(\mu)\, e^{-i\left(z-\tau+\tilde{S}(z,\tau;\mu)\right)}$. This time the uniformity statement follows from the bounds analogous to (22) which are available for $\hat{S}$ in $\mathcal{R}_\rho^{(0)}$ (see Sec. 3.2) and thus for $\hat{\psi} = f_0^{[i]}(\mu)\, e^{i\tau}\,(-i\hat{S} - \frac{1}{2!}\hat{S}^{*2} + \frac{i}{3!}\hat{S}^{*3} + \dots)$ (this sum of repeated convolutions is easy to bound in the main sheet). $\square$

2.3 Proof of Theorem 1

2.3.1 The equation for $\hat{F}$

The formal series $\tilde{\phi}_0 \in z^{-1}\mathcal{P}[[z^{-1}]]$ was defined in Lemma 1. To study its Borel transform, which is the unique solution in $\mathcal{P}[[\zeta]]$ of (17), we first apply a couple of transformations to replace Eq. (17) by another one which lends itself better to the method of "majorants" we have in mind.

Lemma 4 *The formal series $\hat{F}$ defined by Eq. (19) is, for each $\mu \in \mathbb{C}$, the unique solution in $\mathcal{P}[[\zeta]]$ of*

$$\hat{F} = \partial_\zeta^2 \mathcal{E}(\frac{1}{2}\zeta \sin\tau) + \frac{1}{2}\mu\big(\mathcal{E}''(\zeta * \hat{F}^{*2}) + 2\mathcal{E}'(1 * \hat{F}^{*2}) + \mathcal{E}(\hat{F}^{*2})\big), \tag{37}$$

where the operator $\mathcal{E}$ of $\mathcal{P}[[\zeta]]$ is defined using Fourier expansions $\hat{X} = \sum_{k\in\mathbb{Z}} \hat{X}^{[k]}\, e^{ik\tau}$, $\hat{Y} = \sum_{k\in\mathbb{Z}} \hat{Y}^{[k]}\, e^{ik\tau}$ (here $\hat{X}^{[k]}, \hat{Y}^{[k]} \in \mathbb{C}[[\zeta]]$): $\hat{X} = \mathcal{E}\hat{Y}$ iff

$$\hat{X}^{[k]} = \frac{\zeta}{\zeta - ik}\hat{Y}^{[k]}, \qquad k \in \mathbb{Z},$$

and $\mathcal{E}'$, $\mathcal{E}''$ are defined analogously by

$$(\mathcal{E}'\hat{Y})^{[k]} = -\frac{ik}{(\zeta - ik)^2}\hat{Y}^{[k]}, \quad (\mathcal{E}''\hat{Y})^{[k]} = \frac{2ik}{(\zeta - ik)^3}\hat{Y}^{[k]}, \qquad k \in \mathbb{Z}.$$

Proof. The change of unknown series

$$\tilde{\phi}_0 = 4z^{-1} + 4\mu\partial_z^{-1}\tilde{G}, \quad \tilde{G} \in z^{-3}\mathcal{P}[[z^{-1}]]$$

(where we consider ∂_z as an invertible operator $z^{-1}\mathcal{P}[[z^{-1}]] \to z^{-2}\mathcal{P}[[z^{-1}]]$) transforms Eq. (1) into

$$(\partial_\tau + \partial_z)\tilde{G} = -z^{-3}\sin\tau + \frac{1}{2}\mu\,\partial_z\big((z\tilde{G})^2\big).$$

Correspondingly, in the Borel plane, $\hat{\phi}_0 = 4 - 4\mu\zeta^{-1}\hat{G}$, and we can look for $\hat{G}$ as the unique formal solution in $\zeta^2\mathcal{P}[[\zeta]]$ of the equation

$$\zeta^{-1}(\zeta - \partial_\tau)\hat{G} = \frac{1}{2}\zeta\sin\tau + \frac{1}{2}\mu(\partial_\zeta\hat{G})^{*2}. \tag{38}$$

The inverse of the linear operator which appears in the left-hand side is nothing but $\mathcal{E}$:

$$\zeta^{-1}(\zeta - \partial_\tau)\hat{X} = \hat{Y} \Leftrightarrow \hat{X} = \mathcal{E}\hat{Y} \tag{39}$$

(notice that $\hat{X}^{[0]} = \hat{Y}^{[0]}$). Equation (38) can thus be rewritten

$$\hat{G} = \mathcal{E}\big(\frac{1}{2}\zeta\sin\tau\big) + \frac{1}{2}\mu\,\mathcal{E}\big((\partial_\zeta\hat{G})^{*2}\big).$$

Since $\hat{G} \in \zeta^2\mathcal{P}[[\zeta]]$ can be written $\hat{G} = \zeta * \partial_\zeta^2\hat{G}$, we obtain the equation for $\hat{F}$ by differentiating twice: $\hat{F} = \partial_\zeta^2\hat{G}$ is the unique solution in $\mathcal{P}[[\zeta]]$ of an equation which involves the operator $\partial_\zeta^2 \circ \mathcal{E} = \mathcal{E}'' + 2\mathcal{E}' \circ \partial_\zeta + \mathcal{E} \circ \partial_\zeta^2$. Using

$$(\partial_\zeta\hat{G})^{*2} = \zeta * \hat{F}^{*2}, \quad \partial_\zeta\big((\partial_\zeta\hat{G})^{*2}\big) = 1 * \hat{F}^{*2}, \quad \partial_\zeta^2\big((\partial_\zeta\hat{G})^{*2}\big) = \hat{F}^{*2},$$

we see that the resulting equation amounts to (37). $\square$

2.3.2 Analyticity in the main sheet

The formal series $\hat{F}$ can be written $\sum_{j\geq 0} D_j(\tau;\mu)\zeta^j$, where the D_j's are trigonometric polynomials in τ which depend polynomially on μ. We shall now prove the convergence of this series and the holomorphy in $\mathcal{R}^{(0)} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$ of the resulting analytic function; this obviously implies similar results for $\hat{\phi}_0$.

Let us fix $\rho \in]0,1[$. Using the identification between ζ and $\dot{\zeta}$ for points of $\mathcal{R}^{(0)}$, we can rephrase (18) as

$$\mathcal{R}_\rho^{(0)} = \{\, \zeta \in \mathbb{C} \mid \forall \zeta' \in [0,\zeta],\ |\zeta' \pm i| \geq \rho \,\}$$

(see the left part of Figure 1).

Proposition 1 *For each τ and μ, the power series $\hat{F}$ has positive radius of convergence with respect to ζ and the resulting holomorphic function extends analytically to $\mathcal{R}_\rho^{(0)} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$ where it satisfies the inequality (20) with $\ell(\zeta) = |\zeta|$.*

The rest of the present section is devoted to the proof of this proposition. To this end, it is convenient to expand (37) in powers of μ: we find $\hat{F} = \sum_{n\geq 0} \mu^n \hat{F}_n(\zeta,\tau)$ with

$$\hat{F}_0 = \partial_\zeta^2 \mathcal{E}(\frac{1}{2}\zeta \sin\tau) = -\frac{i}{2(\zeta+i)^3} e^{-i\tau} + \frac{i}{2(\zeta-i)^3} e^{i\tau} \tag{40}$$

$$\hat{F}_n = \frac{1}{2} \sum_{n_1+n_2=n-1} \big(\mathcal{E}''(\zeta * \hat{F}_{n_1} * \hat{F}_{n_2}) + 2\mathcal{E}'(1 * \hat{F}_{n_1} * \hat{F}_{n_2}) + \mathcal{E}(\hat{F}_{n_1} * \hat{F}_{n_2})\big), \qquad n \geq 1. \tag{41}$$

Since $\frac{\zeta^k}{k!} * \frac{\zeta^j}{j!} = \frac{\zeta^{k+j+1}}{(k+j+1)!}$ and the operators $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ do not decrease the valuation, we obtain $\hat{F}_n \in \zeta^n \mathcal{P}[[\zeta]]$ (this corresponds to the fact that the coefficients D_j have at most degree n as polynomials in μ). Thus, for each $\mu \in \mathbb{C}$, the series $\sum_{n\geq 0} \mu^n \hat{F}_n(\zeta,\tau)$ converges formally towards $\hat{F}$ in $\mathcal{P}[[\zeta]]$.

The series $\hat{F}_n$ are convergent and it is easy to check by induction on n that the resulting functions are holomorphic in $\mathcal{R}^{(0)} \times (\mathbb{C}/2\pi\mathbb{Z})$. In fact $\hat{F}_0$ is meromorphic, and analyticity in $\mathcal{R}^{(0)}$ is preserved by the operators $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ and the convolution, but the repeated convolutions are responsible for more complicated singularities on $\pm i[1,\infty[$: on the one hand they create higher harmonics and under the action of $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ poles will appear at all points of $i\mathbb{Z}^*$; on the other hand they will create ramification at these singular points. (This is the origin of the Riemann surface $\mathcal{R}$.)

To prove Proposition 1, it is thus sufficient to study the convergence of the series of holomorphic functions $\sum \mu^n \hat{F}_n$.

Definition 1 Majorant Fourier series for $\mathcal{R}^{(0)}$: *we shall write $\hat{A} \ll \hat{\mathcal{A}}$ if*

- *$\hat{A} = \hat{A}(\zeta,\tau) = \sum_{k\in\mathbb{Z}} \hat{A}^{[k]}(\zeta)\, e^{ik\tau}$, where each $\hat{A}^{[k]}$ is analytic in $\mathcal{R}_\rho^{(0)}$;*
- *$\hat{\mathcal{A}} = \hat{\mathcal{A}}(\zeta,\tau) = \sum_{k\in\mathbb{Z}} \hat{\mathcal{A}}^{[k]}(\zeta)\, e^{ik\tau}$, where each $\hat{\mathcal{A}}^{[k]}$ is continuous in $\mathbb{R}^+$ and the Fourier series $\sum \hat{\mathcal{A}}^{[k]}(\zeta)\, e^{ik\tau}$ is convergent for $\tau \in \mathbb{C}/2\pi\mathbb{Z}$ (uniformly for ζ in any compact of $\mathbb{R}^+$);*
- *$\forall k \in \mathbb{Z}$, $\forall \zeta \in \mathcal{R}_\rho^{(0)}$, $|\hat{A}^{[k]}(\zeta)| \leq \hat{\mathcal{A}}^{[k]}(|\zeta|)$.*

Notice that

$$\hat{A} \ll \hat{\mathcal{A}} \quad \Rightarrow \quad \forall \zeta \in \mathcal{R}_\rho^{(0)},\ \forall \tau \in \mathbb{C}/2\pi\mathbb{Z},\ |\hat{A}(\zeta,\tau)| \leq \hat{\mathcal{A}}(|\zeta|, i\,\Im\mathrm{m}\,\tau). \tag{42}$$

Lemma 5 *If $\hat{A} \ll \hat{\mathcal{A}}$ and $\hat{B} \ll \hat{\mathcal{B}}$,*

$$\hat{A} * \hat{B} \ll \hat{\mathcal{A}} * \hat{\mathcal{B}}, \quad \mathcal{E}\hat{A} \ll (1+\rho^{-1})\hat{\mathcal{A}}, \quad \mathcal{E}'\hat{A} \ll \rho^{-2}\hat{\mathcal{A}}, \quad \mathcal{E}''\hat{A} \ll 2\rho^{-3}\hat{\mathcal{A}}.$$

Proof. Analyticity in $\mathcal{R}_\rho^{(0)}$ is preserved under convolution because this set is star-shaped with respect to $\zeta = 0$; the first inequality follows easily. For the other ones, observe that, for all $k \in \mathbb{Z}$ and $\zeta \in \mathcal{R}_\rho^{(0)}$, $|\zeta - ik| \geq \rho|k|$, thus

$$\left|\frac{\zeta}{\zeta - ik}\right| = \left|1 + \frac{ik}{\zeta - ik}\right| \leq 1 + \rho^{-1}, \quad \left|\frac{ik}{(\zeta-ik)^2}\right| \leq \rho^{-2}, \quad \left|\frac{2ik}{(\zeta-ik)^3}\right| \leq 2\rho^{-3}.$$

$\square$

Lemma 6 *We have $\hat{F}_n \ll \hat{\mathcal{F}}_n$ for all $n \geq 0$, where*

$$\hat{\mathcal{F}}_0 = \rho^{-3} \cos\tau,$$
$$\hat{\mathcal{F}}_n = \sum_{n_1+n_2=n-1} (\rho^{-3}\zeta * \hat{\mathcal{F}}_{n_1} * \hat{\mathcal{F}}_{n_2} + \rho^{-2}\, 1 * \hat{\mathcal{F}}_{n_1} * \hat{\mathcal{F}}_{n_2} + \rho^{-1}\hat{\mathcal{F}}_{n_1} * \hat{\mathcal{F}}_{n_2}), \qquad n \geq 1.$$

Moreover, $\hat{\mathcal{F}}_n(\zeta,\tau) = 4^n r_n \rho^{-3n-2}\hat{P}_n(\zeta)\cos^{n+1}\tau$ where $0 < r_n \leq 1$ and the $\hat{P}_n$'s are polynomials satisfying

$$\forall \zeta, X \geq 0, \quad \sum_{n\geq 0} X^n \hat{P}_n(\zeta) \leq 2\rho^{-1}\, e^{\kappa(X)\zeta}, \qquad \text{with } \kappa(X) = \rho^{-1}\max\big(6X, (6X)^{1/3}\big). \tag{43}$$

Proof. In view of the induction Formulae (40) and (41) and of the previous lemma (using $\rho^{-1} \geq 1$), it is clear that the $\hat{\mathcal{F}}_n$'s are majorant Fourier series for the $\hat{F}_n$'s. Their formal Laplace transforms $\tilde{\mathcal{F}}_n(z,\tau)$ are easy to compute, because the generating series $\tilde{\mathcal{F}} = \sum_{n\geq 0}\mu^n\tilde{\mathcal{F}}_n$ satisfies the quadratic equation

$$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_0 + \mu(\rho^{-1} + \rho^{-2}z^{-1} + \rho^{-3}z^{-2})\tilde{\mathcal{F}}^2, \qquad \tilde{\mathcal{F}}_0 = \rho^{-3}z^{-1}\cos\tau,$$

hence

$$\tilde{\mathcal{F}} = \rho^{-3}z^{-1}(\cos\tau)\mathsf{R}\big(4\mu\rho^{-3}(\rho^{-1}z^{-1} + \rho^{-2}z^{-2} + \rho^{-3}z^{-3})\cos\tau\big), \tag{44}$$

with $\mathsf{R}(x) = 2x^{-1}\big(1-(1-x)^{1/2}\big)$. Since $\mathsf{R}(x) = \sum_{n\geq 0} r_n x^n$ with $0 < r_n \leq 1$, we end up with

$$\tilde{\mathcal{F}}_n = 4^n r_n \rho^{-3n-2}\tilde{P}_n(z)\cos^{n+1}\tau, \quad \text{with} \quad \tilde{P}_n(z) = (\rho z)^{-1}\big((\rho z)^{-1} + (\rho z)^{-2} + (\rho z)^{-3}\big)^n,$$

and the corresponding formula for $\hat{\mathcal{F}}_n$ involves $\hat{P}_n = \mathcal{B}\tilde{P}_n$.

Letting $\hat{P}_n(\zeta) = \sum_{j\geq 0} P_{n,j}\frac{\zeta^j}{j!}$, we can write

$$\sum_{n\geq 0} X^n \hat{P}_n(\zeta) = \sum_{j\geq 0} Q_j(X)\frac{\zeta^j}{j!}, \qquad \text{with } Q_j(X) = \sum_{n\geq 0} P_{n,j}X^n.$$

Now,

$$\tilde{Q}(z,X) = \sum_{j\geq 0} Q_j(X) z^{-j-1} = \sum_{n\geq 0} X^n\tilde{P}_n(z) = \frac{(\rho z)^{-1}}{1 - X\big((\rho z)^{-1} + (\rho z)^{-2} + (\rho z)^{-3}\big)}$$

has positive radius of convergence with respect to z^{-1} for all $X \geq 0$. Observing that

$$\max\Big(|\rho z|^{-1}, |\rho z|^{-3}\Big) \leq \frac{1}{6X} \quad \Rightarrow \quad |z\tilde{Q}(z,X)| \leq 2\rho^{-1}$$

and using the Cauchy inequalities, we obtain $Q_j(X) \leq 2\rho^{-1}\kappa(X)^j$, which yields (43). $\square$

In view of (42), Lemma 6 implies

$$\forall \zeta \in \mathcal{R}_\rho^{(0)},\ \forall \tau \in \mathbb{C}/2\pi\mathbb{Z}, \quad |\hat{F}_n(\zeta,\tau)| \le 4^n \rho^{-3n-2} \hat{P}_n(|\zeta|) \cosh^{n+1}(\Im\mathrm{m}\,\tau) \tag{45}$$

for each n. With (43), this is sufficient to obtain the analyticity in $\mathcal{R}^{(0)}$ of $\hat{F}$ for all τ and μ, and a bound

$$|\hat{F}(\zeta,\tau;\mu)| \le 2\rho^{-3} \cosh(\Im\mathrm{m}\,\tau)\, e^{\kappa(X)|\zeta|}, \qquad \text{with } X = 4\rho^{-3}|\mu| \cosh(\Im\mathrm{m}\,\tau),$$

which yields (20) (with $\ell(\zeta) = |\zeta|$).

This ends the proof of Proposition 1.

2.3.3 Analytic continuation in the nearby sheets

We now give the precise definition of the sets $\mathcal{R}_\rho^{(1)}$ introduced in Sec. 2.1.2 and complete the proof of Theorem 1. Let $\rho \in \left]0, \frac{1}{3}\right[$. We follow [GS01, p. 535–539], but it is simpler to fix the integer parameter which was called M there to its maximal value:

$$M = \left[\frac{1}{2}(\rho^{-1} - 1)\right].$$

Thus $M \ge 1$ and the discs $D_m = D(|m|i, m\rho)$ with $-M-1 \le m \le M+1$ do not overlap; let

$$\dot{\mathcal{R}}_\rho = \mathbb{C} \setminus \Big(\bigcup_{\substack{-M-1 \le m \le M+1 \\ m \ne 0}} D_m \Big).$$

(Mark the use of $M+1$ here instead of M in [GS01]: we seize the opportunity to correct this misprint.) We shall sometimes use the convention $D_0 = \{0\}$.

Definition 2 *We call $\mathcal{R}_\rho^{(1)}$ the subset of $\mathcal{R}$ consisting of all the points ζ which can be represented by a path contained in $\dot{\mathcal{R}}_\rho$ and such that the shortest such path γ_ζ is either*

1. *a straight segment;*
2. *or the union of a straight segment issuing from the origin and tangent to D_m, with $-M \le m \le M$ and $m \ne 0$, and of an arc of the circle ∂D_m ending at $\dot{\zeta}$, and we require in that situation that this arc of circle be shorter than a half-circle*[4] *and that the backward half-tangent $L(\zeta)$ to γ_ζ at $\dot{\zeta}$ do not meet $D_{m\pm 1}$;*
3. *or the union of a straight segment issuing from the origin and tangent to D_m, with $-M \le m \le M$ and $m \ne 0$, of an arc of the circle ∂D_m and of a straight segment $S(\zeta)$ tangent to D_m, ending at $\dot{\zeta}$ and such that the half-line $L(\zeta)$ which extends $S(\zeta)$ backwards from $\dot{\zeta}$ do not meet $D_{m\pm1}$; we also require in that situation that the arc of circle be shorter than a half-circle. See Figure 4.*

There is in fact a certain amount of latitude in the definition of a set like $\mathcal{R}_\rho^{(1)}$. The point is to cover $\mathcal{R}^{(1)}$ as ρ tends to 0 and to control the symmetrically contractile paths Γ_ζ

[4]This condition is just a way of ensuring the necessary property that $\zeta_1 \in \mathcal{R}_\rho^{(1)}$ whenever $\zeta_1 \in \gamma_\zeta$; unfortunately it had been omitted in [GS01], as noticed by A. Fruchard whom we thank.

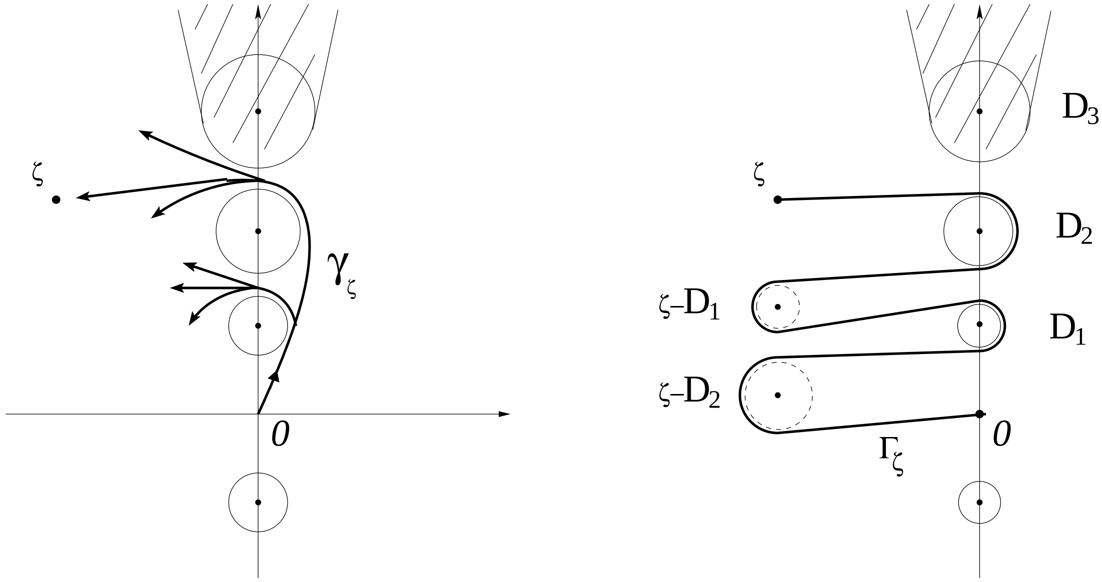

Figure 4: The paths γ_ζ and Γ_ζ define the same point $\zeta \in \mathcal{R}_\rho^{(1)} \subset \mathcal{R}$

associated with the paths γ_ζ well enough to guarantee the stability under convolution of the property of being holomorphic in $\mathcal{R}_\rho^{(1)}$.

Indeed, when trying to follow the analytic continuation of a convolution product along the path γ_ζ for a given point ζ of $\mathcal{R}_\rho^{(1)}$, one is led to introduce the path Γ_ζ defined as follows (see Figure 4):

- in case 1 above, Γ_ζ coincide with γ_ζ;
- in case 3, if $m \geq 1$, Γ_ζ is a union of straight segments and arcs of circle obtained as the shortest path which starts from the origin, meanders between the discs $\zeta - D_m, D_1, \ldots, \zeta - D_{m-k}, D_{k+1}, \ldots, \zeta - D_1, D_m$ (in that order) and then reaches ζ (if $m \leq -1$, replace D_{m-k} by D_{m+k} and D_{k+1} by D_{-k-1} in the previous sentence);
- in case 2, the description is the same except that there is no straight segment from D_k to $\zeta - D_{m-k}$ $(0 \leq k \leq m)$ because of tangencies.

One can check that Γ_ζ is contained in $\mathcal{R}_\rho^{(1)}$ and is homotopic to γ_ζ, i.e. it defines the same point ζ of the Riemann surface. Moreover Γ_ζ is symmetrically contratile, i.e. it is symmetric with respect to its midpoint and can be deformed continuously into the trivial path $\{0\}$ using only symmetric paths issuing from 0 and contained in $\mathcal{R}_\rho^{(1)}$.

As a consequence, when two holomorphic functions $\hat{A}$ and $\hat{B}$ are given on $\mathcal{R}_\rho^{(1)}$, their convolution product too extends analytically to $\mathcal{R}_\rho^{(1)}$. Its analytic continuation is indeed given explicitly by the formula

$$\hat{A} * \hat{B}(\zeta) = \int_{\Gamma_\zeta} \hat{A}(\zeta_1)\hat{B}(\zeta_2)\, d\zeta_1,$$

where ζ_2 is determined as the symmetric point of ζ_1 with respect to the midpoint $\zeta/2$ of Γ_ζ.

To some extent, this formula makes it possible to obtain bounds for the convolution products in $\mathcal{R}_\rho^{(1)}$. We reproduce here the corresponding lemma from [GS01] (Lemma 9, p. 538):

Lemma 7 *Let $\ell(\zeta)$ denote the length of the path Γ_ζ for any $\zeta \in \mathcal{R}_\rho^{(1)}$. If $\hat{A}$ and $\hat{B}$ are holomorphic functions in $\mathcal{R}_\rho^{(1)}$ which satisfy*

$$\forall \zeta \in \mathcal{R}_\rho^{(1)}, \quad |\hat{A}(\zeta)| \le \hat{\mathcal{A}}\big(\ell(\zeta)\big) \quad \text{and} \quad |\hat{B}(\zeta)| \le \hat{\mathcal{B}}\big(\ell(\zeta)\big),$$

where $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are continuous non-decreasing functions on $\mathbb{R}^+$, their convolution product is holomorphic in $\mathcal{R}_\rho^{(1)}$ and satisfies

$$\forall \zeta \in \mathcal{R}_\rho^{(1)}, \quad |\hat{A} * \hat{B}(\zeta)| \le \hat{\mathcal{A}} * \hat{\mathcal{B}}\big(\ell(\zeta)\big).$$

For the proof the reader is referred to [GS01, p. 539]. This proof uses crucially the fact that each ζ_1 on Γ_ζ has a curvilinear abscissa not smaller than $\ell(\zeta_1)$; this can be checked directly with our definition of Γ_ζ, thanks to the limitation we have imposed on the possible paths γ_ζ when defining $\mathcal{R}_\rho^{(1)}$. Variants of Definition 2 are conceivable, but one must take this point into account.

With these preliminaries, we can easily adapt the work of the previous section to obtain the analyticity of $\hat{F}$ in $\mathcal{R}_\rho^{(1)}$ and the inequality (20) with the function ℓ defined in Lemma 7.

Definition 3 Majorant Fourier series for $\mathcal{R}^{(1)}$: *we shall write $\hat{A} \ll_1 \hat{\mathcal{A}}$ if*

- $\hat{A} = \hat{A}(\zeta,\tau) = \sum_{k\in\mathbb{Z}} \hat{A}^{[k]}(\zeta)\, e^{ik\tau}$, *where each $\hat{A}^{[k]}$ is analytic in $\mathcal{R}_\rho^{(1)}$;*
- $\hat{\mathcal{A}} = \hat{\mathcal{A}}(\zeta,\tau) = \sum_{k\in\mathbb{Z}} \hat{\mathcal{A}}^{[k]}(\zeta)\, e^{ik\tau}$, *where each $\hat{\mathcal{A}}^{[k]}$ is continuous and non-decreasing in $\mathbb{R}^+$, and the Fourier series $\sum \hat{\mathcal{A}}^{[k]}(\zeta)\, e^{ik\tau}$ is convergent for $\tau \in \mathbb{C}/2\pi\mathbb{Z}$ (uniformly for ζ in any compact of $\mathbb{R}^+$);*
- $\forall k \in \mathbb{Z}$, $\forall \zeta \in \mathcal{R}_\rho^{(1)}$, $|\hat{A}^{[k]}(\zeta)| \le \hat{\mathcal{A}}^{[k]}\big(\ell(\zeta)\big)$.

In the present section, the majorant Fourier series $\hat{\mathcal{A}}(\zeta,\tau)$ will be always trigonometric polynomials, but this will not be the case any longer in Sec. 3.2 where we use again Definition 3.

An obvious adaptation of Lemma 5 which incorporates Lemma 7 yields

Lemma 8 *We have $\hat{F}_n \ll_1 \hat{\mathcal{F}}_n$ for all $n \ge 0$, with the same majorants $\hat{\mathcal{F}}_n$ as in Lemma 6.*

The property (42) being replaced by

$$\hat{A} \ll_1 \hat{\mathcal{A}} \quad \Rightarrow \quad \forall \zeta \in \mathcal{R}_\rho^{(1)},\ \forall \tau \in \mathbb{C}/2\pi\mathbb{Z},\ |\hat{A}(\zeta,\tau)| \le \hat{\mathcal{A}}\big(\ell(\zeta), i\,\Im\mathrm{m}\,\tau\big), \tag{46}$$

we now see that, for $\zeta \in \mathcal{R}_\rho^{(1)}$, $\tau \in \mathbb{C}/2\pi\mathbb{Z}$, $n \ge 0$,

$$|\hat{F}_n(\zeta,\tau)| \le \rho^{-2} \cosh(\Im\mathrm{m}\,\tau)\, X^n \hat{P}_n\big(\ell(\zeta)\big), \qquad X = 4\rho^{-3}\cosh(\Im\mathrm{m}\,\tau),$$

and we conclude like before by using (43).

To end the proof of Theorem 1, we just need to check the validity of (21) for any point ζ of $\mathcal{R}_\rho^{(1)}$. For this we use our description of Γ_ζ in Case 3 of Definition 2 when $m \geq 1$ (with obvious adaptations for the other cases): each of the $m+1$ segments between D_k and $\zeta - D_{m-k}$ $(0 \leq k \leq m)$ has length $\leq |\overset{\bullet}{\zeta} - mi| + m\rho$, each of the m segments between $\zeta - D_{m-k}$ and D_{k+1} $(0 \leq k \leq m-1)$ has length $\leq |\overset{\bullet}{\zeta} - (m+1)i| + (m+1)\rho$, the arcs of circle have total length $\leq 2(1+2+\ldots+m)2\pi\rho$, hence $\ell(\zeta) \leq (2m+1)\rho + 2m(m+1)(1+\rho) + 2m(m+1)\rho\pi$.

2.4 Proof of Theorem 2

We could begin the proof of Theorem 2 by verifying inductively that, near $\zeta = i$, all the components of the Taylor expansion with respect to μ of $\hat{\phi}_0$ and thus $\hat{\phi}_0$ itself[5] have the form of a simple pole plus a logarithmic term as announced in (29). This would correspond to the approach adopted in [GS01] at this stage. But there is a more concise method, which relies on the concepts of "major" and alien derivation.

2.4.1 Majors of singularities

We follow here [Eca92b, Sec. 2.1] and [CNP93b, "Pré II"], and describe the basic notions related to singularities of holomorphic functions.

Using a suitable translation, its is sufficient to deal with singularities at the origin of the Riemann surface of the logarithm $\underset{\bullet}{\mathbb{C}} = \{\zeta = r\, e^{i\theta},\ \theta \in \mathbb{R},\ r > 0\}$. Given two real numbers $\theta_1 < \theta_2$, we shall denote by S_{θ_1,θ_2} the sector of $\underset{\bullet}{\mathbb{C}}$ defined by $\theta_1 < \arg\zeta < \theta_2$.

Definition 4 *Let $\theta \in \mathbb{R}$ and $\alpha > 0$. Consider the space of germs of holomorphic functions $\overset{\vee}{\varphi}(\zeta)$ defined for $\zeta \in S_{\theta-\alpha-2\pi,\theta+\alpha}$ and $|\zeta|$ small enough. Its quotient by the space $\mathbb{C}\{\zeta\}$ of regular germs is, by definition, the space $\mathrm{SING}_{\theta,\alpha}$ of singularities in the direction θ with aperture 2α. A germ $\overset{\vee}{\varphi}$ is called a major, its class in $\mathrm{SING}_{\theta,\alpha}$ is called the singularity of $\overset{\vee}{\varphi}(\zeta)$ and is denoted by $\mathrm{sing}(\overset{\vee}{\varphi})$ or $\overset{\triangledown}{\varphi}$.*

To any singularity $\overset{\triangledown}{\varphi}$ in $\mathrm{SING}_{\theta,\alpha}$ we associate its minor $\hat{\varphi}$, which is obtained from any major $\overset{\vee}{\varphi}$ by the formula

$$\hat{\varphi}(\zeta) = \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(\zeta\, e^{-2\pi i}).$$

It is thus a germ of holomorphic function in $S_{\theta-\alpha,\theta+\alpha}$.

A singularity and its minor are also called a "microfunction" and its "variation". The simplest examples of singularities are $\delta = \mathrm{sing}\left(\frac{1}{2\pi i\zeta}\right)$, or more generally $\delta^{(n)} = \mathrm{sing}\left(\frac{(-1)^n n!}{2\pi i\zeta^{n+1}}\right)$ if $n \in \mathbb{N}$, and $\mathrm{sing}\left(\hat{\varphi}(\zeta)\frac{\log\zeta}{2\pi i}\right)$ for any $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ (the chosen determination

[5]In fact such a verification is best performed on the components $\hat{F}_n$ of the auxiliary function $\hat{F}$, like in Sec. 2.3, and we would obtain for them a polar part of order 3. Notice however that the argument for the convergence with respect to μ of the polar parts and of the logarithmic terms which is given in [GS01] (at the end of Sec. 4.1) is somewhat incomplete: as remarked by E. Delabaere whom we thank, the convergence of the series of functions $\sum \mu^n \hat{F}_n$ for $\zeta \in \mathcal{R}_\rho^{(1)}$ is not sufficient because $\mathcal{R}_\rho^{(1)}$ does not contain a path which encircles the singular point; this can easily be remedied by a slight modification of the definition of $\mathcal{R}_\rho^{(1)}$ (the paths γ_ζ must be authorized to turn around $D(\pm i,\rho)$ until a second crossing of the imaginary axis; this does not alter much the shape of the corresponding paths Γ_ζ described in Sec. 2.3.3).

of the logarithm does not matter); these are elements of $\mathrm{SING}_{\theta,\alpha}$ for all θ and α. The last example is a particular case of an "integrable singularity".

Definition 5 *An integrable minor is a germ of holomorphic function $\hat{\varphi}$ in $S_{\theta-\alpha,\theta+\alpha}$ which admits a primitive $\hat{\psi}$ such that $\hat{\psi}(\zeta) \to 0$ as $\zeta \to 0$ uniformly in any proper subsector of $S_{\theta-\alpha,\theta+\alpha}$. We denote by $\mathrm{ANA}^{\mathrm{int}}_{\theta,\alpha}$ the corresponding space of germs.*

A singularity is said to be integrable if it admits a major $\overset{\vee}{\varphi}$ such that $\zeta\overset{\vee}{\varphi}(\zeta) \to 0$ uniformly as $\zeta \to 0$ in any proper subsector of $S_{\theta-\alpha-2\pi,\theta+\alpha}$. The space of integrable singularities in the direction θ with aperture 2α is denoted by $\mathrm{SING}^{\mathrm{int}}_{\theta,\alpha}$.

Integrable singularities are also called "small" singularities or microfunctions.

Lemma 9 *The linear map $\overset{\triangledown}{\varphi} \mapsto \hat{\varphi}$ induces a bijection from $\mathrm{SING}^{\mathrm{int}}_{\theta,\alpha}$ onto $\mathrm{ANA}^{\mathrm{int}}_{\theta,\alpha}$. The inverse map is denoted $\hat{\varphi} \mapsto {}^b\hat{\varphi}$.*

For example, if $\hat{\varphi} \in \mathbb{C}\{\zeta\}$, we have ${}^b\hat{\varphi} = \mathrm{sing}\left(\hat{\varphi}(\zeta)\frac{\log\zeta}{2\pi i}\right)$; for the proof of the lemma in the general case, one can use a Cauchy integral.

Formula (16) turns $\mathrm{ANA}^{\mathrm{int}}_{\theta,\alpha}$ into an algebra: this is the so-called convolution of minors. The convolution law is extended as follows:

Lemma 10 *Suppose that $\overset{\vee}{\varphi}$ and $\overset{\vee}{\psi}$ are the majors of two singularities $\overset{\triangledown}{\varphi}$ and $\overset{\triangledown}{\psi}$ of $\mathrm{SING}_{\theta,\alpha}$, and that $u \in S_{\theta-\alpha-2\pi,\theta+\alpha}$ has sufficiently small modulus. The germ defined by*

$$\overset{\vee}{\chi}_u(\zeta) = \int_{I_{u,\zeta}} \overset{\vee}{\varphi}(\zeta_1)\overset{\vee}{\psi}(\zeta-\zeta_1)\,d\zeta_1, \qquad \arg u - \pi < \arg\zeta < \arg u, \quad |\zeta| \textit{ small enough,}$$

where $I_{u,\zeta}$ is the straight segment $[u, u\,e^{-i\pi}+\zeta]$, extends analytically to $S_{\theta-\alpha-2\pi,\theta+\alpha}$ (for $|\zeta|$ small enough); its class in $\mathrm{SING}_{\theta,\alpha}$ does not depend on u and depends only on $\overset{\triangledown}{\varphi}$ and $\overset{\triangledown}{\psi}$. The law $\mathrm{sing}(\overset{\vee}{\chi}_u) = \overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ turns $\mathrm{SING}_{\theta,\alpha}$ into a commutative algebra, with unit δ.

If moreover $\overset{\triangledown}{\varphi}$ and $\overset{\triangledown}{\psi}$ are integrable singularities, so is $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$, and the induced law on $\mathrm{SING}^{\mathrm{int}}_{\theta,\alpha}$ is the counterpart of the convolution of minors: ${}^b\hat{\varphi} * {}^b\hat{\psi} = {}^b(\hat{\varphi} * \hat{\psi})$.

For any integrable singularity $\overset{\triangledown}{\varphi}$, one can define the convolutive analogue of the exponential:

$$\exp_*(\overset{\triangledown}{\varphi}) = \delta + \overset{\triangledown}{\varphi} + \frac{1}{2!}\overset{\triangledown}{\varphi} * \overset{\triangledown}{\varphi} + \frac{1}{3!}\overset{\triangledown}{\varphi} * \overset{\triangledown}{\varphi} * \overset{\triangledown}{\varphi} + \dots \tag{47}$$

The convergence of the corresponding series of minors is particularly obvious when $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ (see [CNP93b, p. 161–162] for the general case) and, in this case,

$$\frac{1}{2\pi i\zeta} + {}^b(e^{\hat{\varphi}} - 1) \tag{48}$$

is a major of this singularity.

Remark 6 The regular germs $\hat{\phi}_0(\zeta)$ and $\hat{F}(\zeta)$ of which Theorem 1 asserts the existence (considering the variables τ and μ as parameters), can be viewed as integrable singularities: $\overset{\triangledown}{\phi}_0 = {}^b\hat{\phi}_0$, $\overset{\triangledown}{F} = {}^b\hat{F}$. But the above notions will be relevant only when studying the singularities of their analytic continuation at any $\omega \in i\mathbb{Z}^*$. For instance, if we choose the direction $\theta = \pi/2$, the analyticity of $\hat{\phi}_0$ in $\mathcal{R}^{(1)}$ allows us to consider

$$\overset{\triangledown}{X} = \operatorname{sing}\big(\hat{\phi}_0(i+\zeta)\big) \tag{49}$$

as an element of $\mathrm{SING}_{\theta,\alpha}$ for any $\alpha \le \pi$. Observe that this means that we consider here the translation $\zeta \mapsto \zeta' = i + \zeta$ as an automorphism between the part of $\underset{\bullet}{\mathbb{C}}$ where $\theta - 2\pi < \arg\zeta < \theta$ and $|\zeta| < 1$ and the part of $\mathcal{R}^{(0)}$ where $\overset{\bullet}{\zeta'} \notin i\,[1,+\infty[$ and $|\overset{\bullet}{\zeta'} - i| < 1$, whereas the points ζ' in the half-sheet of $\mathcal{R}$ accessed from $\mathcal{R}^{(0)}$ by crossing $]i, 2i[$ from the right to the left correspond to $\theta < \arg\zeta < \theta + \alpha$ (and the half-sheet accessed by crossing from the left to the right corresponds to $\theta - 2\pi - \alpha < \arg\zeta < \theta - 2\pi$).

It should be clear at this point, by (47) and (48), that Theorem 2 amounts exactly to the existence of a scalar $f_0^{[i]}(\mu)$ such that

$$\overset{\triangledown}{X} = f_0^{[i]}(\mu)\, e^{i\tau} \exp_*\big(-{}^{b}(i\hat{S})\big). \tag{50}$$

Remark 7 Among general singularities, some can be expanded in series of monomials, i.e. elementary singularities like $\delta^{(n)}$ with $n \in \mathbb{Z}$ (we define $\delta^{(-n)} = {}^{b}\big(\zeta^{n-1}/\Gamma(n)\big)$ if $n \ge 1$), or $\operatorname{sing}\big((1-e^{-2\pi i\sigma})^{-1}\zeta^{\sigma-1}/\Gamma(\sigma)\big)$ with $\sigma \in \mathbb{C}\setminus\mathbb{Z}$, etc. For them one can define the *formal Laplace transform* by its action on monomials (the image is z^{-n} for the first example above, $z^{-\sigma}$ for the second) in such a way that convolution and multiplication are exchanged. We then recover the formal Borel transform defined in Sec. 2.1.1 as the inverse map (taking the minors in the case of integrable singularities). The differentiation with respect to z corresponds to the operator

$$\partial \,:\, \operatorname{sing}\big(\overset{\vee}{\varphi}(\zeta)\big) \mapsto \operatorname{sing}\big(-\zeta\overset{\vee}{\varphi}(\zeta)\big),$$

which is a derivation of the algebra of singularities.

However it would not be convenient to restrict our attention to this kind of singularities only. For instance our chain of reasoning in Sec. 2.4.3 will not presuppose that $\overset{\triangledown}{X}$ admits such an expansion.

2.4.2 First alien derivations

Resurgence theory focuses on singularities whose minors enjoy good properties of analytic continuation, but we shall formulate these properties very progressively like in [GS01, Sec. 5.4] (because our proof that $\hat{\phi}_0$ fulfills such properties is progressive).

We now formulate a definition adapted to situations in which the first singular point in the direction θ is $\omega = e^{i\theta}$, since we are interested in such a case (with $\theta = \pi/2$ or $3\pi/2$).

Definition 6 *Let $\theta \in \mathbb{R}$ and $\beta > 0$. We define $\mathrm{RES}^{(1)}_{\theta,\beta}$ to be the space of all $\overset{\triangledown}{\varphi}$ which belong to $\mathrm{SING}_{\theta,\alpha}$ for some $\alpha > 0$ and whose minor $\hat{\varphi}$ extends analytically along $\big]0, e^{i\theta}\big[$ so that the germ $\overset{\vee}{\psi}(\zeta) = \hat{\varphi}(e^{i\theta} + \zeta)$, which is thus defined*[6] *for $\arg\zeta$ close to $\theta - \pi$, extends analytically to $S_{\theta-2\pi-\beta,\theta+\beta}$ (for $|\zeta|$ small enough).*

We define the operator $\Delta_{e^{i\theta}} : \mathrm{RES}^{(1)}_{\theta,\beta} \to \mathrm{SING}_{\theta,\beta}$ by $\Delta_{e^{i\theta}}\overset{\triangledown}{\varphi} = \operatorname{sing}(\overset{\vee}{\psi})$.

[6] Observe that we are choosing the same lift of the translation $\zeta \mapsto e^{i\theta} + \zeta$ as in Remark 6 when $\theta = \pi/2$.

It is easy to check that

$$[\Delta_{e^{i\theta}}, \partial] = -e^{i\theta} \Delta_{e^{i\theta}}, \tag{51}$$

where ∂ is the "natural" derivation mentioned in Remark 7. The operator $\Delta_{e^{i\theta}}$ is called the *alien derivation of index* $e^{i\theta}$ because of

Proposition 2 *Whenever* $\overset{\triangledown}{\varphi}_1, \overset{\triangledown}{\varphi}_2 \in \mathrm{RES}^{(1)}_{\theta,\beta}$*, we have* $\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2 \in \mathrm{RES}^{(1)}_{\theta,\beta}$ *and*

$$\Delta_{e^{i\theta}}(\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2) = (\Delta_{e^{i\theta}} \overset{\triangledown}{\varphi}_1) * \overset{\triangledown}{\varphi}_2 + \overset{\triangledown}{\varphi}_1 * (\Delta_{e^{i\theta}} \overset{\triangledown}{\varphi}_2). \tag{52}$$

According to Theorem 1, we have $\overset{\triangledown}{\phi}_0 \in \mathrm{RES}^{(1)}_{\theta,\beta}$ with $\theta = \pi/2$ for any $\beta \le \pi$, and Formula (49) can be rephrased as

$$\overset{\triangledown}{X} = \Delta_i \overset{\triangledown}{\phi}_0, \qquad \overset{\triangledown}{\phi}_0 = {}^b\hat{\phi}_0.$$

Of course, all the definitions of Sec. 2.4.1 and 2.4.2 are extended to the case of majors depending on further variables $\tau \in \mathbb{C}/2\pi\mathbb{Z}$, $\mu \in \mathbb{C}$, by treating them as parameters.

When there is a formal counterpart as indicated in Remark 7 (and this will be the case for $\overset{\triangledown}{\phi}_0$ and $\overset{\triangledown}{X}$), the operator $\Delta_{e^{i\theta}}$ can be viewed as acting on formal expansions in z; this allows us to formulate everything in the formal model, as we did in the comments following Theorem 2 in Sec. 2.1.4, although things are to be interpreted in the convolutive model. The relation (51) can then be expressed as the commutation of $\frac{\partial}{\partial z}$ and the *dotted alien derivation* $\overset{\bullet}{\Delta}_\omega = e^{-\omega z} \Delta_\omega$ (here $\omega = e^{i\theta}$). On the other hand, when dealing with singularities admitting majors depending analytically on $\tau \in \mathbb{C}/2\pi\mathbb{Z}$, the operators $\frac{\partial}{\partial \tau}$ and Δ_ω commute (and so do $\frac{\partial}{\partial \tau}$ and $\overset{\bullet}{\Delta}_\omega$).

2.4.3 The major of the singularity of $\hat{\phi}_0$ at i

We now move on to the proof of the existence of $f_0^{[i]}(\mu)$ such that the relation (50) holds, where $\overset{\triangledown}{X} = \Delta_i \overset{\triangledown}{\phi}_0$. The underlying idea is that $\overset{\bullet}{\Delta}_i$ commutes with ∂_z and ∂_τ, thus $\overset{\bullet}{\Delta}_i \tilde{\phi}_0$ satisfies the linearization of Eq. (1) around $\tilde{\phi}_0$, i.e. the same equation (23) as $z - \tau + \tilde{S}$; but any solution of (23) must be a "function" of $z - \tau + \tilde{S}$, and in the case of $\overset{\bullet}{\Delta}_i \tilde{\phi}_0$ the requirement of periodicity in τ and the nature of its dependence with respect to z will force it to be proportional to $e^{-iz+i\tau-i\tilde{S}}$. We present below rigorous arguments which use the language of majors and singularities.

Since its formal Laplace transform is solution of Eq. (1), $\overset{\triangledown}{\phi}_0$ satisfies

$$\partial_\tau \overset{\triangledown}{\phi}_0 - \frac{1}{8} \delta^{(2)} * \big(\partial \overset{\triangledown}{\phi}_0\big)^{*2} + 2\delta^{(-2)}(1 - \mu \sin\tau) = 0.$$

Applying Δ_i to this equation, using (52) and (51), we find

$$\partial_\tau \overset{\triangledown}{X} + \overset{\triangledown}{D}_0 * \partial \overset{\triangledown}{X} = i \overset{\triangledown}{D}_0 * \overset{\triangledown}{X}, \tag{53}$$

where $\overset{\triangledown}{D}_0 = -\frac{1}{4}\delta^{(2)} * \big(\partial \overset{\triangledown}{\phi}_0\big) = \delta - \mu^b\hat{F}$ in view of (25), i.e. $\overset{\triangledown}{D}_0$ corresponds to the formal series $\tilde{D}_0 = 1 - \mu\tilde{F}$.

Eq. (23) reads $\partial_\tau \tilde{Y} + \tilde{D}_0(z,\tau;\mu)\partial_z \tilde{Y} = 0$ and Lemma 3 said that $\tilde{Y} = z - \tau + \tilde{S}$ satisfies it. Moreover, Prop. 5 in Sec. 3.2 will show that $\tilde{S}$ admits a formal Borel transform $\hat{S} \in \mathbb{C}\{\zeta\}$ and then $\overset{\triangledown}{S} = {}^b\hat{S}$. If we take any function of $\tilde{Y}$, e.g. by $e^{iz - i\tau + i\tilde{S}}$, it will also verify Eq. (23). Thus the formal series $\tilde{Z} = e^{-i\tau + i\tilde{S}}$ satisfies

$$\partial_\tau \tilde{Z} + \tilde{D}_0 \partial_z - \tilde{Z} = -i\tilde{D}_0 \tilde{Z},$$

and $\overset{\triangledown}{Z} = e^{-i\tau} \exp_*(i{}^b\hat{S})$ satisfies an equation analogous to (53) but with opposite right-hand side.

As a consequence, $\overset{\triangledown}{\varphi} = \overset{\triangledown}{X} * \overset{\triangledown}{Z} = e^{-i\tau}\overset{\triangledown}{X} * \exp_*(i{}^b\hat{S})$ verifies

$$\partial_\tau \overset{\triangledown}{\varphi} + \overset{\triangledown}{D}_0 * \partial\overset{\triangledown}{\varphi} = 0. \tag{54}$$

But $\overset{\triangledown}{\varphi}$ admits a major $\overset{\vee}{\varphi}(\zeta,\tau;\mu)$ which is holomorphic for $|\zeta| < 1$, $-5\pi/2 < \arg\zeta < 3\pi/2$, and $(\tau,\mu) \in (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$, since this is the case for $\overset{\triangledown}{X}$ by virtue of Theorem 1 and for ${}^b\hat{S}$ according to Prop. 5, and $\overset{\triangledown}{D}_0$ admits such a major $\overset{\vee}{D}_0$ too (for instance $\overset{\vee}{D}_0(\zeta,\tau;\mu) = \frac{1}{2\pi i\zeta} - \mu\hat{F}(\zeta,\tau;\mu)\frac{log\zeta}{2\pi i}$). We can thus expand Eq. (54) in Fourier-Taylor series with respect to τ and μ.

Writing $\overset{\vee}{\varphi}(\zeta,\tau;\mu) = \sum_{n\geq 0}\mu^n \overset{\vee}{\varphi}_n(\zeta,\tau) = \sum_{n\geq 0,\ k\in\mathbb{Z}} \mu^n\, e^{ik\tau}\overset{\vee}{\varphi}_{n,k}(\zeta)$, and $\overset{\triangledown}{\varphi} = \sum_{n\geq 0}\mu^n\overset{\triangledown}{\varphi}_n$, $\overset{\triangledown}{D}_0 = \delta - \sum_{n\geq 0}\mu^{n+1}\overset{\triangledown}{F}_n(\zeta,\tau)$, where $\overset{\triangledown}{F}_n = {}^b\hat{F}_n$ and $\hat{F}_n$ are given in (40) and (41), we find:

$$\partial_\tau \overset{\triangledown}{\varphi}_0 + \partial\overset{\triangledown}{\varphi}_0 = 0 \tag{55}$$

$$\partial_\tau \overset{\triangledown}{\varphi}_n + \partial\overset{\triangledown}{\varphi}_n = \sum_{n_1+n_2=n-1} \overset{\triangledown}{F}_{n_1} * \partial\overset{\triangledown}{\varphi}_{n_2}, \qquad n \geq 1. \tag{56}$$

Eq. (55) amounts to $(ik - \zeta)\overset{\vee}{\varphi}_{0,k}(\zeta) \in \mathbb{C}\{\zeta\}$ for each $k \in \mathbb{Z}$. This yields the existence of $A_0 \in \mathbb{C}$ such that

$$\overset{\vee}{\varphi}_{0,0}(\zeta) - \frac{A_0}{2\pi i\zeta} \in \mathbb{C}\{\zeta\},$$

while $\overset{\vee}{\varphi}_{0,k}(\zeta) \in \mathbb{C}\{\zeta\}$ for $k \neq 0$. Hence $\overset{\triangledown}{\varphi}_0 = A_0\delta$ and $\partial\overset{\triangledown}{\varphi}_0 = 0$. Inserting this into (56), we find successively $\overset{\triangledown}{\varphi}_1 = A_1\delta, \overset{\triangledown}{\varphi}_2 = A_2\delta, \ldots$ by the same argument, for some sequence of complex numbers $A_1, A_2, \ldots$

The expansion of $\overset{\triangledown}{\varphi}$ thus reduces to $A(\mu)\delta$, where the series $A(\mu) = \sum_{n\geq 0} A_n\mu^n$ defines an entire function because of the domain of holomorphy which was known in advance for the aforementioned major $\overset{\vee}{\varphi}(\zeta,\tau;\mu)$.

The relation (15) expresses the symmetry of $\tilde{\phi}_0$ with respect to the involution $(\tau,\mu) \mapsto (\tau+\pi, -\mu)$, a property which is shared by $\tilde{S}$ and thus by $e^{i\tau}\overset{\triangledown}{\varphi} = \Delta_i\overset{\triangledown}{\phi}_0 * \exp_*(i{}^b\hat{S})$; the oddness of $f_0^{[i]}(\mu) = A(\mu)$ follows. The value of A_1 is easily computed from (40). This ends the proof of Theorem 2.

Remark 8 The uniform estimate (22) shows that $f_0^{[i]}(\mu)$ is an entire function and has exponential growth with respect to μ.

3 Formal integral and Bridge equation

3.1 Statement of the results

3.1.1 Huygens principle

In the resurgent approach, it is important not to restrict to one particular formal solution of the problem in hand. For a system of ordinary differential equations for instance, one uses the notion of "formal integral": a formal solution which depends on the appropriate number of free parameters; where it converges, one could describe locally all possible solutions by varying these parameters. For a first-order partial differential equation like (1), there is a classical notion of "complete solution", of which we shall use the formal counterpart.

Following [Che02], for an equation $f\left(q, \frac{\partial\phi}{\partial q}, \phi\right) = 0$ in a region of $\mathbb{R}^{2n+1}$ where at least one of the partial derivatives $\frac{\partial f}{\partial p_i}(q,p,s)$ does not vanish, one defines a (local) complete solution to be a function $\phi(q,\alpha)$ on some open subset of $\mathbb{R}^{2n}$ such that

$$(q,\alpha) \mapsto \left(q, \frac{\partial\phi}{\partial q}(q,\alpha), \phi(q,\alpha)\right)$$

is a (local) parameterization of the hypersurface $f = 0$ in the space of 1-jets $\mathbb{R}^{2n+1}$. Huygens principle asserts that, locally, any solution of the equation can be obtained from ϕ by formation of *envelope*.

In our case, the equation can be written

$$H(z,\tau,\partial_z\phi,\partial_\tau\phi) = 0,$$

with a Hamiltonian function $H(z,\tau,C,E) = E - \frac{1}{8}z^2C^2 + 2z^{-2}(1-\mu\sin\tau)$. One can check that, if $\phi(z,\tau,c)$ solves the equation for each c (where c denotes a one-dimensional parameter) and if

$$z \mapsto \partial_c\phi(z,\tau,c)$$

is invertible for each τ and c, the function $(z,\tau,c,a) \mapsto a + \phi(z,\tau,c)$ (where a denotes another one-dimensional variable) is a complete solution.

A way of checking the validity of Huygens principle in that case is to consider the exact-symplectic transformation $\mathcal{T}_\phi : (z',\tau',c,e) \mapsto (z,\tau,C,E)$ generated by the function $(z,\tau,c,e) \mapsto e\tau + \phi(z,\tau,c)$, i.e. implicitly defined by

$$\begin{aligned} z' &= \partial_c\phi(z,\tau,c), & C &= \partial_z\phi(z,\tau,c), \\ \tau' &= \tau, & E &= e + \partial_\tau\phi(z,\tau,c). \end{aligned}$$

Indeed, in the coordinates (z',τ',c,e), the Hamiltonian reduces to e and the corresponding Hamilton-Jacobi equation is trivially solved; but the formulae relating the solutions in different canonical systems of coordinates are not so easy to write down, since they are the graphs of their differentials which correspond one to the other by the symplectic transformation $\mathcal{T}_\phi$ (they must represent the same Lagrangian manifold contained in the zero energy level).

This explains why we can content ourselves with looking for a formal solution as described in (5): $\partial_c\tilde{\phi}(z,\tau,c) = z - \tau + \dots$ will be formally invertible.

In fact, to obtain the Bridge equation, we shall make use of the Huygens principle only at an infinitesimal level, i.e. of the fact that the solutions of the linearization of Eq. (1) around the formal integral $\tilde{\phi}(z,\tau,c)$ are functions of $\partial_c\tilde{\phi}$. But we hope to study in a further article the possibility of resuming the formal integral as we did for its first term $\tilde{\phi}_0$ and to investigate the connection formulae between its two resumations, $\phi^+(z,\tau,c)$ and $\phi^-(z,\tau,c)$.

3.1.2 The components $\tilde{\phi}_n$ of the formal integral

A series $\tilde{\phi}(z,\tau,c;\mu) = \sum_{n\geq 0} c^n \tilde{\phi}_n(z,\tau;\mu)$ beginning with $\tilde{\phi}_0(z,\tau;\mu)$ defined in Lemma 1 solves formally Eq. (1) if and only if

$$\begin{aligned} \big(\partial_\tau + \tilde{D}_0\partial_z\big)\tilde{\phi}_1 &= 0 && (57)\\ \big(\partial_\tau + \tilde{D}_0\partial_z\big)\tilde{\phi}_n &= \frac{1}{8}z^2 \sum_{\substack{n_1+n_2=n\\ n_1,n_2\geq 1}} \partial_z\tilde{\phi}_{n_1}\partial_z\tilde{\phi}_{n_2}, \qquad n\geq 2, && (58)\end{aligned}$$

where $\tilde{D}_0(z,\tau;\mu) = 1-\mu\tilde{F}(z,\tau;\mu) = -\frac{1}{4}z^2\partial_z\tilde{\phi}_0(z,\tau;\mu)$. Observe that Eq. (57) is nothing but Eq. (23); as already mentioned we choose the solution $\tilde{\phi}_1 = z-\tau+\tilde{S}$ defined in Lemma 3. We shall now determine a sequence of formal series $\{\tilde{\phi}_n(z,\tau;\mu)\}_{n\geq 2}$ which satisfies (58) and which gives rise to the formal integral (5) announced in the introduction.

Technically, it will be convenient to consider $x = z+\tilde{S}(z,\tau;\mu)$ as a formal change of variable which *straightens* the vector field $\partial_\tau + \tilde{D}_0(z,\tau;\mu)\partial_z$.

From now on, during the proofs, we omit the dependence on μ to avoid cumbersome notations.

Lemma 11 *For each $\mu\in\mathbb{C}$, the relation*

$$x = z+\tilde{S}(z,\tau;\mu) \Leftrightarrow z = x+\tilde{R}(x,\tau;\mu) \tag{59}$$

defines a formal series $\tilde{R}\in x^{-1}\mathcal{P}[[x^{-1}]]$, which is the unique solution in that space of the equation

$$\tilde{R}(x,\tau;\mu) = -\tilde{S}(x+\tilde{R}(x,\tau;\mu),\tau;\mu). \tag{60}$$

It is also the unique solution in $x^{-1}\mathcal{P}[[x^{-1}]]$ of the equation

$$(\partial_\tau+\partial_x)\tilde{R}(x,\tau;\mu) = -\mu\tilde{F}(x+\tilde{R}(x,\tau;\mu),\tau;\mu), \tag{61}$$

which expresses the conjugation of $\partial_\tau+\tilde{D}_0(z,\tau;\mu)\partial_z$ to the straightened vector field $\partial_\tau+\partial_x$. Like $\tilde{\phi}_0$ and $\tilde{S}$, the series $\tilde{R}$ is antisymmetric with respect to the involution σ : $(z,\tau)\mapsto(-z,\pi-\tau)$.

Remark 9 Eq. (59) expresses that two formal transformations $x=\Phi(z,\tau;\mu)$ and $z=\Psi(x,\tau;\mu)$ are mutually inverse; Eq. (60) amounts to the relation $\Phi\circ\Psi = \mathrm{Id}$. But one can also define $\tilde{R}$ through (61) and recover the series $\tilde{S}$ as the unique solution in $z^{-1}\mathcal{P}[[z^{-1}]]$ of the equation

$$\tilde{S}(z,\tau;\mu) = -\tilde{R}(z+\tilde{S}(z,\tau;\mu),\tau;\mu) \tag{62}$$

which expresses the relation $\Psi\circ\Phi = \mathrm{Id}$. This alternative definition of $\tilde{S}$ will be useful for the study of its Borel transform.

Proof. The equivalence of (59) and (60) (or (62)) is clear, and it is easy to check the existence of a unique solution $\tilde{R}$, which is antisymmetric.

The conjugation between the vector fields is equivalent to $1+\partial_x\tilde{R}+\partial_\tau\tilde{R}=\tilde{D}_0(x+\tilde{R},\tau)$, which amounts to (61), but also to $1=\tilde{D}_0(z,\tau)(1+\partial_z\tilde{S})+\partial_\tau\tilde{S}$, which amounts to $(\partial_\tau+\tilde{D}_0\partial_z)(z-\tau+\tilde{S})=0$. The conclusion thus follows from Lemma 3. □

Proposition 3 *For each $\mu\in\mathbb{C}$ and $n\geq 2$, there exists a unique series $\tilde{\phi}_n$ in $\mathcal{P}[z][[z^{-1}]]$ whose constant term (the coefficient of $e^{ik\tau}z^{-j}$ when $k=j=0$) vanishes, such that any solution of Eq. (58) in that space is sum of $\tilde{\phi}_n$ and an arbitrary complex number. The series $\tilde{\phi}_n$ itself is antisymmetric with respect to the involution $\sigma:(z,\tau)\mapsto(-z,\pi-\tau)$ and can be written*

$$\tilde{\phi}_n(z,\tau;\mu)=P_n(z,\tau;\mu)+\tilde{\phi}_{[n]}(z,\tau;\mu),\qquad P_n\in\mathcal{P}[z],\quad \tilde{\phi}_{[n]}\in z^{-1}\mathcal{P}[[z^{-1}]],\tag{63}$$

where P_n has degree $2n-1$ in z.

The sequence $\{\tilde{\phi}_n\}_{n\geq 2}$ is determined inductively by Formulae (65), (66) and (67) below. We supplement it by the first two terms $\tilde{\phi}_0$ and $\tilde{\phi}_1=z-\tau+\tilde{S}$ defined by Lemmas 1 and 3, so as to obtain the "formal integral" (or "formal complete solution")

$$\tilde{\phi}(z,\tau,c;\mu)=\sum_{n\geq 0}c^n\tilde{\phi}_n(z,\tau;\mu).$$

Remark 10 Antisymmetry is a convenient feature of our problem which stems from our choice of the perturbation $2(1-\mu\sin\tau)$ in (1). It will guarantee that the right-hand sides in (58) have zero mean residuums (i.e. the τ-average of their z-residuum, or the coefficient of $e^{ik\tau}z^{-j}$ when $k=0$ and $j=1$, vanishes) and admit therefore primitives with respect to z which belong to $\mathcal{P}[z][[z^{-1}]]$.

With other choices of perturbation, it can be necessary to admit multiples of $\log z$: the components $\tilde{\phi}_n$ belong to $\mathcal{P}[z][[z^{-1}]]\oplus\mathbb{C}\log z$ (despite the non-linearity of the equation these logarithmic terms do not proliferate, because the right-hand sides in (58) are built from the derivatives of the components and still belong to $\mathcal{P}[z][[z^{-1}]]$).

Proof. Let us perform the change of variable (59) in the system of equations (57), (58): in view of Lemma 11, the equations for the new unknown series $\tilde{g}_n(x,\tau)=\tilde{\phi}_n(x+\tilde{R}(x,\tau),\tau)$ read

$$(\partial_\tau+\partial_x)\,\tilde{g}_1 = 0,\tag{64}$$

$$(\partial_\tau+\partial_x)\,\tilde{g}_n = \tilde{B}_n,\qquad n\geq 2,\tag{65}$$

with

$$\tilde{B}_n(x,\tau)=\frac{1}{8}\,x^2\left(\frac{1+x^{-1}\tilde{R}}{1+\partial_x\tilde{R}}\right)^2\sum_{\substack{n_1+n_2=n\\ n_1,n_2\geq 1}}\partial_x\tilde{g}_{n_1}\partial_x\tilde{g}_{n_2}.\tag{66}$$

We have already chosen $\tilde{\phi}_1(z,\tau)=z-\tau+\tilde{S}(z,\tau)$, which corresponds to the solution

$$\tilde{g}_1(x,\tau)=x-\tau$$

of (64). Observe that the only solutions of this homogeneous equation in $\mathcal{P}[x][[x^{-1}]]$ are the constants. Hence the first claim of the proposition amounts to the existence of a unique solution $\tilde{g}_n(x,\tau)$ of (65) in $\mathcal{P}[x][[x^{-1}]]$ with zero constant term for each $n \geq 2$.

Given $B \in \mathcal{P}[x][[x^{-1}]]$, the equation $(\partial_\tau + \partial_x)A = B$ is easily studied by expanding both sides first in Fourier series and then in power series of x. For each $k \in \mathbb{Z}$, we find $(ik + \partial_x)A^{[k]} = B^{[k]}$ which admits obviously a unique solution in $\mathbb{C}[x][[x^{-1}]]$ if $k \neq 0$:

$$A^{[k]} = \frac{1}{ik}\left(1 + \frac{1}{ik}\partial_x\right)^{-1} B^{[k]}.$$

The only possible obstruction is the residuum in $B^{[0]}$ (coefficient of x^{-1}); when this residuum vanishes, the equation $\partial_x A^{[0]} = B^{[0]}$ admits a unique solution in $\mathbb{C}[x][[x^{-1}]]$ without constant term. Moreover the valuation with respect to x^{-1} is increased by 1 at most when passing from $B^{[k]}$ to $A^{[k]}$, therefore the obtained Fourier series $\sum A^{[k]}(x)\, e^{ik\tau}$ belongs to $\mathcal{P}[x][[x^{-1}]]$ too.

Since in a series which is symmetric with respect to σ the Fourier coefficient of zero index is even in x, there is no obstruction when B is symmetric, and the corresponding solution is antisymmetric in that case.

One thus proceeds by induction on $n \geq 2$: the series $\partial_x \tilde{g}_m(x,\tau)$ being symmetric with respect to σ for $1 \leq m \leq n-1$ and having a polynomial part in x of degree $2m-2$, the series $\tilde{B}_n$ $(n \geq 2)$ is symmetric and has a polynomial part of degree $2n-2$, thus the corresponding solution $\tilde{g}_n$ is antisymmetric and has a polynomial part of degree $2n-1$.

We recover the solutions $\tilde{\phi}_n$ of Eq. (58) by

$$\tilde{\phi}_n(z,\tau;\mu) = \tilde{g}_n(z + \tilde{S}(z,\tau),\tau;\mu) \tag{67}$$

and they are antisymmetric with respect to the involution σ, since our formal change of variable commutes with σ. Moreover, as $\tilde{S} \in z^{-1}\mathcal{P}[[z^{-1}]]$, the degree of the polynomial parts for $\tilde{\phi}_n$ and $\tilde{g}_n$ is the same. $\square$

3.1.3 Resurgent properties of the formal integral

We shall see that the formal Borel transforms $\hat{\phi}_n(\zeta,\tau;\mu)$ of the series $\tilde{\phi}_{[n]}$ introduced in (63) are convergent for ζ close to the origin, including in the case of $\tilde{\phi}_{[1]} = \tilde{S}$ (we set $P_1 = z - \tau$). According to Remark 7, we may thus consider all the components $\tilde{\phi}_n$ of the formal integral (for fixed τ and μ) as the formal counterparts of singularities

$$\overset{\triangledown}{\phi}_n = \overset{\triangledown}{P}_n + {}^b\hat{\phi}_n, \tag{68}$$

where $\overset{\triangledown}{P}_n$ is a linear combination of the elementary singularities $\delta^{(j)} = \operatorname{sing}\left(\frac{(-1)^j j!}{2\pi i \zeta^{j+1}}\right)$ with $0 \leq j \leq 2n-1$ corresponding to the monomials z^j (we set $P_0 = 0$).

In order to state our main result on the analytic structure of these germs $\hat{\phi}_n$, we resume the description of the basic concepts of Resurgence theory that has started in Sec. 2.4.1 and 2.4.2.

Definition 7 *We define* RES *to be the space of all* $\overset{\triangledown}{\varphi}$ *which belong to* $\mathrm{SING}_{\theta,\alpha}$ *for some* θ,α *and whose minor* $\hat{\varphi}$ *extends analytically to the universal cover of* $\mathbb{C}\setminus i\mathbb{Z}$. *Its elements* $\overset{\triangledown}{\varphi}$ *are called resurgent functions (with singularities above* $i\mathbb{Z}$*).*

Notice that we have restricted to $i\mathbb{Z}$ the set of possible singular points for the minors whereas the general theory can handle much richer singular sets. We could have imposed the further restriction that the minors be regular at the origin, since this will be the case for the $\hat{\phi}_n$'s, but this does not facilitate particularly the exposition.

Definition 8 *Let $\omega = m\,e^{i\theta} \in \underset{\bullet}{\mathbb{C}}$ with $\theta \in \frac{\pi}{2} + \pi\mathbb{Z}$ and $m \in \mathbb{N}^*$. We define a linear operator Δ_ω : RES $\to$ RES by the formula*

$$\Delta_\omega \overset{\triangledown}{\varphi} = \sum_{\varepsilon_1,\dots,\varepsilon_{m-1}\in\{+,-\}} \frac{p(\varepsilon)!q(\varepsilon)!}{m!} \operatorname{sing}\big(\hat{\varphi}^{\varepsilon_1,\dots,\varepsilon_{m-1}}(\omega+\zeta)\big), \tag{69}$$

where $p(\varepsilon)$ and $q(\varepsilon) = m-1-p(\varepsilon)$ denote the numbers of signs '$+$' and of signs '$-$' in the sequence ε, and $\hat{\varphi}^{\varepsilon_1,\dots,\varepsilon_{m-1}}(\omega+\zeta)$ denotes the germ which is obtained for $\arg\zeta$ *close to $\theta-\pi$ by following the analytic continuation of the minor $\hat{\varphi}$ along $]0,\omega[$ and circumventing the intermediary singular points $re^{i\theta}$ to the right if $\varepsilon_r = +$ and to the left if $\varepsilon_r = -$.*

This definition is obviously compatible with Definition 6 when $m = 1$. The operators Δ_ω are called *alien derivations of index* ω because of

Proposition 4 *The space* RES *is a subalgebra of* $\mathrm{RES}^{(1)}_{\theta,\beta}$ *for any $\theta \in \frac{\pi}{2} + \pi\mathbb{Z}$ and $\beta > 0$; the operators Δ_ω satisfy the Leibniz rule.*

With our definitions, the operators Δ_ω and $\Delta_{\omega e^{2\pi i}}$ differ (see [CNP93b], p. 208–209), but their action coincide on resurgent functions whose minors are regular at the origin (these minors are analytic on the Riemann surface $\mathcal{R}$ of Sec. 2.1.2).

We have $[\Delta_\omega, \partial] = -\omega\Delta_\omega$, i.e. the *dotted alien derivations* $\dot{\Delta}_\omega = e^{-\omega z}\,\Delta_\omega$ commute with ∂, as was the case for $m = 1$. As already mentioned at the end of Sec. 2.4.2, it is convenient to view the operators Δ_ω as acting in the formal model whenever the resurgent functions in hand admit a formal counterpart.

For instance, if $\overset{\triangledown}{\varphi} \in$ RES admits a formal Laplace transform $\tilde{\varphi} \in \mathcal{P}[z][[z^{-1}]]$ (which means that $\overset{\triangledown}{\varphi}$ can be decomposed like $\overset{\triangledown}{\phi}_n$ in (68), or equivalently that its minor $\hat{\varphi}$ is regular at the origin), saying that $\Delta_{\omega_r}\dots\Delta_{\omega_1}\overset{\triangledown}{\varphi} \in \mathcal{P}[z][[z^{-1}]]$ for all $\omega_1,\dots,\omega_r$ amounts to the property that all the singularities encountered when following the analytic continuation of $\hat{\varphi}$ are sums of polar parts (linear combinations of $\delta^{(j)}$) and logarithmic singularities (integrable singularities with regular minors). Such a resurgent function ($\overset{\triangledown}{\varphi}$ or $\tilde{\varphi}$) is said to be *simply ramified.*

One must keep in mind that $\Delta_{\omega_r}\dots\Delta_{\omega_1}\overset{\triangledown}{\varphi}$ is a combination of the singularities of various determinations of $\hat{\varphi}$ at the point $\omega_1+\dots+\omega_r$, and that the knowledge of all these successive alien derivatives allows one to compute the whole singular behavior of $\hat{\varphi}$.

Theorem 3 *All the components $\tilde{\phi}_n$ $(n \geq 0)$ of the formal integral (as defined in Proposition 3) are simply ramified resurgent functions depending analytically on $\tau \in \mathbb{C}/2\pi\mathbb{Z}$ and $\mu \in \mathbb{C}$, whose minors have at most exponential growth along the non-vertical half-lines contained in $\mathcal{R}$.*

For each $\omega \in i\mathbb{Z}^*$, there exists a sequence of entire functions $\big\{f_n^{[\omega]}(\mu)\big\}_{n\geq 0}$ (where $f_0^{[i]}$ coincides with the function introduced in Theorem 2) such that the "Bridge equation" *holds:*

$$\dot{\Delta}_\omega \tilde{\phi} = f^{[\omega]} e^{-\omega \partial_c \tilde{\phi}}, \qquad f^{[\omega]} = \sum_{n \geq 0} f_n^{[\omega]}(\mu) c^n. \tag{70}$$

This equation must be understood as a system of infinitely many "resurgence relations" obtained by expanding it in powers of c: $\Delta_\omega \tilde{\phi}_0 = f_0^{[\omega]} e^{\omega\tau} e^{-\omega\tilde{S}}$ *and*

$$\Delta_\omega \tilde{\phi}_n = \Big[f_n^{[\omega]} + \sum_{r=1}^{n} \frac{(-1)^r \omega^r}{r!} \sum_{\substack{n_0 + n_1 + \ldots + n_r = n + r \\ n_0 \geq 0,\ n_1, \ldots, n_r \geq 2}} n_1 \ldots n_r \, f_{n_0}^{[\omega]} \, \tilde{\phi}_{n_1} \ldots \tilde{\phi}_{n_r} \Big] e^{\omega\tau} e^{-\omega\tilde{S}}$$

for $n \geq 1$*.*

Observe that the alien derivative $\Delta_\omega \tilde{\phi}_n$ is thus determined in terms of the scalars $f_0^{[\omega]}(\mu), \ldots, f_n^{[\omega]}(\mu)$ and of the series $\tilde{\phi}_1, \ldots, \tilde{\phi}_{n+1}$. The successive alien derivatives of the components are determined by applying one or several dotted alien derivatives to both sides of the Bridge equation and expanding the result in powers of c. For that, one just needs to know the rule for the alien derivative of an exponential, which is deduced from the Leibnitz rule applied to (47). For instance,

$$\begin{aligned} \dot{\Delta}_{\omega_2} \, \dot{\Delta}_{\omega_1} \tilde{\phi} &= f^{[\omega_1]} \, \dot{\Delta}_{\omega_2} e^{-\omega_1 \partial_c \tilde{\phi}} = -\omega_1 f^{[\omega_1]} e^{-\omega_1 \partial_c \tilde{\phi}} \partial_c \, \dot{\Delta}_{\omega_2} \tilde{\phi} \\ &= \omega_1 f^{[\omega_1]} \Big(\omega_2 f^{[\omega_2]} \partial_c^2 \tilde{\phi} - \partial_c f^{[\omega_2]} \Big) e^{-(\omega_1 + \omega_2) \partial_c \tilde{\phi}} \end{aligned}$$

allows one to compute each series $\dot{\Delta}_{\omega_2} \, \dot{\Delta}_{\omega_1} \tilde{\phi}_n$ in terms of $f_0^{[\omega_1]}, \ldots, f_n^{[\omega_1]}$, $f_0^{[\omega_2]}, \ldots, f_{n+1}^{[\omega_2]}$ and $\tilde{\phi}_1, \ldots, \tilde{\phi}_{n+2}$.

Theorem 3 does not make any assertion about the convergence of the formal integral $\tilde{\phi}$. Then its results must be understood as results for any of the function $\tilde{\phi}_n$. In the case that $\tilde{\phi}(z, \tau, c; \mu)$ was convergent as a series in c, Eq. (70) would gives its alien derivatives in terms of $\partial_c \tilde{\phi}$, and it would allow us to obtain its resumations $\phi^{\pm}$ as we did for $\tilde{\phi}_0$ in Sec. 2.2.

Furthermore, by using Huygens principle, these two resumations will give us two complete solutions of Eq. (1), that should be related. We postpone this study to a further article.

The rest of the article is devoted to the proof of Theorem 3. We begin by proving the analyticity of the auxiliary Borel transforms $\hat{S}$ and $\hat{R}$ for $\zeta \in \mathcal{R}^{(1)}$ by a majorant method which is very similar to that of Sec. 2.3.2 and 2.3.3 (in fact this study could have been placed in Sec. 2; the result on $\hat{S}$ was already used in Sec. 2.2 and in Sec. 2.4.3). Then we shall employ the ideas of Sec. 2.4.3 in a more systematic way to derive progressively the resurgence relations and propagate analyticity from one sheet to the other in the Riemann surface $\mathcal{R}$.

3.2 Study of the auxiliary Borel transforms $\hat{R}$ and $\hat{S}$

3.2.1 Statement of the analyticity results

Proposition 5 *The formal series* $\tilde{R}(x, \tau; \mu) \in x^{-1}\mathcal{P}[[x^{-1}]]$ *defined in Lemma 11 and the formal series* $\tilde{S}(z, \tau; \mu) \in z^{-1}\mathcal{P}[[z^{-1}]]$ *defined in Lemma 3 admit formal Borel transforms* $\hat{R}(\zeta, \tau; \mu)$ *and* $\hat{S}(\zeta, \tau; \mu)$ *which are convergent for* ζ *close to the origin (uniformly* *in τ and μ). The resulting holomorphic functions of the three variables ζ, τ and μ extend analytically to $\mathcal{R}^{(1)} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$.*

Moreover, for each $\rho \in \left]0, \frac{1}{3}\right[$, there exist a non-decreasing function κ and a positive number K such that

$$|\hat{R}(\zeta,\tau;\mu)|,\ |\hat{S}(\zeta,\tau;\mu)| \le K|\mu| \cosh(\Im\mathrm{m}\,\tau)\, e^{\kappa\left(|\mu|\cosh(\Im\mathrm{m}\,\tau)\right)\ell(\zeta)} \tag{71}$$

for $\zeta \in \mathcal{R}^{(1)}_\rho$, $\tau \in \mathbb{C}/2\pi\mathbb{Z}$, $\mu \in \mathbb{C}$, with the same function ℓ as in Theorem 1 (defined in Lemma 7 of Sec. 2.3.3).

The rest of Sec. 3.2 is devoted to the proof of this proposition; we shall omit the dependence on μ of the various series for the sake of clarity. We begin by two lemmas which will be the starting point of a majorant series method.

Lemma 12 *The formal Borel transform of $\tilde{R}$ can be written $\hat{R}(\zeta,\tau;\mu) = \sum_{n\ge 0} \hat{R}_n(\zeta,\tau;\mu)$, with formal series $\hat{R}_n \in \zeta^n\mathcal{P}[[\zeta]]$ defined by the induction formulae*

$$\begin{aligned} \hat{R}_0 &= \mu\,\mathcal{E}_0\hat{F} \\ \hat{R}_n &= \mu\,\mathcal{E}_0\hat{T}_n, \quad \hat{T}_n = \sum_{r=1}^{n} \frac{(-1)^r}{r!}(\zeta^r\hat{F}) * \sum_{n_1+\ldots+n_r=n-r} \hat{R}_{n_1} * \ldots * \hat{R}_{n_r}, \qquad n\ge 1, \end{aligned}$$

where $\hat{F}(\zeta,\tau)$ was defined by (19) and the operator $\mathcal{E}_0$ is defined using Fourier expansions:

$$\hat{A} = \mathcal{E}_0\hat{B} \;\Leftrightarrow\; (\forall k\in\mathbb{Z})\;\; \hat{A}^{[k]}(\zeta) = \frac{1}{\zeta - ik}\hat{B}^{[k]}(\zeta) \tag{72}$$

provided $\hat{B}^{[0]}$ vanishes at $\zeta = 0$.

Proof. Since $\hat{F}$ satisfies the condition $\hat{F}^{[0]}(0) = 0$ (because of (40) and (41)), one can check inductively that, for each n, $\hat{R}_n$ is well defined and belong to $\zeta^n\mathcal{P}[[\zeta]]$. The operator $\mathcal{E}_0$ was defined in such a way that the corresponding formal Laplace transforms satisfy

$$\begin{aligned} -(\partial_\tau+\partial_x)\tilde{R}_0 &= \mu\tilde{F} \\ -(\partial_\tau+\partial_x)\tilde{R}_n &= \mu\tilde{T}_n, \quad \tilde{T}_n = \sum_{r=1}^{n}\frac{1}{r!}(\partial_z^n\tilde{F}) \sum_{n_1+\ldots+n_r=n-r}\tilde{R}_{n_1}\ldots\tilde{R}_{n_r}, \qquad n\ge 1, \end{aligned}$$

which shows, after applying Taylor's formula to $\tilde{F}$, that the series $\sum \tilde{R}_n(x,\tau)$ (which is formally convergent) satisfies Eq. (61). □

Lemma 13 *The formal Borel transform of $\tilde{S}$ can be written $\hat{S}(\zeta,\tau;\mu) = \sum_{n\ge 0}\hat{S}_n(\zeta,\tau;\mu)$, with formal series $\hat{S}_n \in \zeta^n\mathcal{P}[[\zeta]]$ defined by the induction formulae*

$$\begin{aligned} \hat{S}_0 &= -\hat{R} \\ \hat{S}_n &= \sum_{r=1}^{n}\frac{(-1)^{r+1}}{r!}(\zeta^r\hat{R}) * \sum_{n_1+\ldots+n_r=n-r}\hat{S}_{n_1} * \ldots * \hat{S}_{n_r}, \qquad n\ge 1. \end{aligned}$$

Proof. One recognizes the use of the Taylor formula in (62). □

3.2.2 Majorant series for $\hat{R}$

Let us fix $\rho \in \left]0, \frac{1}{3}\right[$. The proof of the analyticity of $\hat{R}$ for $\zeta \in \mathcal{R}_\rho^{(1)}$ will be somewhat similar to the study of $\hat{F}$ in Sec. 2.3.2 and 2.3.3. Using the notation of Definition 3, we can rephrase what we had obtained there: $\hat{F} = \sum_{n\geq 0} \mu^n \hat{F}_n(\zeta,\tau)$ with, according to Lemma 8, $\hat{F}_n \ll_1 \hat{\mathcal{F}}_n$, the majorant series $\hat{\mathcal{F}}_n$ being defined in Lemma 6. We can thus write

$$\hat{F} = \hat{F}_0 + \hat{F}_*, \qquad \hat{F}_*(\zeta,\tau) = \sum_{n\geq 1} \mu^n \hat{F}_n(\zeta,\tau),$$

$$\hat{F}_0 \ll_1 \hat{\mathcal{F}}_0 = \rho^{-3}\cos\tau, \quad \hat{F}_* \ll_1 \hat{\mathcal{F}}_*, \quad \hat{F} \ll_1 \hat{\mathcal{F}} = \hat{\mathcal{F}}_0 + \hat{\mathcal{F}}_*,$$

defining $\hat{\mathcal{F}}_*(\zeta,\tau)$ by its Fourier coefficients $\hat{\mathcal{F}}_*^{[k]}(\zeta) = \sum_{n\geq 1} |\mu|^n \hat{\mathcal{F}}_n^{[k]}(\zeta)$. These coefficients $\hat{\mathcal{F}}_*^{[k]}$ are entire functions of ζ which vanish at 0, with a Taylor expansion involving only non-negative coefficients. By a slight improvement of the reasoning in the proof of Lemma 6, the Fourier series $\sum \hat{\mathcal{F}}_*^{[k]}(\zeta)e^{ik\tau}$ is seen to converge for $\tau \in \mathbb{C}/2\pi\mathbb{Z}$ uniformly for ζ in any compact of $\mathbb{C}$. The formal Laplace transform of $\hat{\mathcal{F}}(\zeta,\tau)$ is given explicitly in (44) (replacing μ by $|\mu|$).

We observe that, in view of the definition of $\mathcal{E}_0$ and of Lemma 7, the series $\hat{R}_n$ introduced in Lemma 12 are convergent and define functions which are analytic in $\mathcal{R}_\rho^{(1)} \times (\mathbb{C}/2\pi\mathbb{Z})$. We shall now look for majorant series $\hat{\mathcal{R}}_n$ for them which will help us to prove the convergence of the series of holomorphic functions $\sum \hat{R}_n$.

To begin, we decompose $\hat{R}_0$ as $\mu\mathcal{E}_0\hat{F}_0 + \mu\mathcal{E}_0\hat{F}_*$, with

$$\mathcal{E}_0\hat{F}_0 \ll_1 \rho^{-1}\hat{\mathcal{F}}_0$$

(because $\hat{\mathcal{F}}_0^{[0]}(\zeta) = 0$ and $|\zeta - ik| \geq \rho$ for $k \neq 0$ and $\zeta \in \mathcal{R}_\rho^{(1)}$). Majorant series for $\mathcal{E}_0\hat{F}_*$ and the remaining $\hat{R}_n$'s will be deduced from

Lemma 14 *There exists $\alpha > 0$ which depends only on ρ such that, whenever $\hat{B} \ll_1 \hat{\mathcal{B}}$ with entire functions $\hat{\mathcal{B}}^{[k]}$ which vanish at 0 and whose Taylor expansion involves only non-negative coefficients,*

$$\hat{A} = \mathcal{E}_0\hat{B} \quad \Rightarrow \quad \hat{A} \ll_1 \hat{\mathcal{A}} = \alpha\partial_\zeta\hat{\mathcal{B}}.$$

Notice that the formal Laplace transforms of the majorant series are related by $\tilde{\mathcal{A}}(z,\tau) = \alpha z\tilde{\mathcal{B}}(z,\tau)$, and that our hypothesis implies that $\tilde{\mathcal{B}}(z,\tau) = O(z^{-2})$.

Proof. The vanishing of the functions $\hat{B}^{[k]}$ at the origin allows us to write $\hat{A} = \zeta^{-1}\hat{C}$ with $\hat{C} = \mathcal{E}\hat{B} \ll_1 2\rho^{-1}\hat{\mathcal{B}}$ according to Lemma 5 adapted to $\mathcal{R}_\rho^{(1)}$. Observing that moreover $|\dot{\zeta}|^{-1}\ell(\zeta)$ is bounded in $\mathcal{R}_\rho^{(1)}$ (because of (21), using $|\dot{\zeta}| = \ell(\zeta)$ on the main sheet and $|\dot{\zeta}| \geq \rho$ on the other ones), we thus have for all $k \in \mathbb{Z}$ and $\zeta \in \mathcal{R}_\rho^{(1)}$,

$$|\hat{A}^{[k]}(\zeta)| \leq \frac{2\rho^{-1}}{|\dot{\zeta}|}\hat{\mathcal{B}}^{[k]}\big(\ell(\zeta)\big) \leq \frac{\bar{\alpha}}{\ell(\zeta)}\hat{\mathcal{B}}^{[k]}\big(\ell(\zeta)\big)$$

for some $\bar{\alpha} > 0$. We conclude by comparing $\hat{\mathcal{B}}^{[k]}(\ell) = \sum_{j\geq 1} b_j^{[k]}\ell^j$ and $\partial_\zeta\hat{\mathcal{B}}^{[k]}(\ell) = \sum_{j\geq 1} j b_j^{[k]}\ell^{j-1}$ for any $\ell \in \mathbb{R}^+$. □

Corollary 2 *We have $\hat{R}_n \ll_1 \hat{\mathcal{R}}_n$ for each n, with majorant series defined by the induction formulae*

$$\hat{\mathcal{R}}_0 = |\mu|\left(\rho^{-1}\hat{\mathcal{F}}_0 + \alpha\partial_\zeta\hat{\mathcal{F}}_*\right)$$

$$\hat{\mathcal{R}}_n = |\mu|\alpha\partial_\zeta\hat{\mathcal{T}}_n, \quad \hat{\mathcal{T}}_n = \sum_{r=1}^{n}\frac{1}{r!}(\zeta^r\hat{\mathcal{F}}) * \sum_{n_1+\ldots+n_r=n-r}\hat{\mathcal{R}}_{n_1} * \ldots * \hat{\mathcal{R}}_{n_r}, \qquad n \geq 1.$$

Proof. One checks by induction on n that $\hat{T}_n \ll_1 \hat{\mathcal{T}}_n$, thanks to the behavior of majorant series with respect to convolution in $\mathcal{R}^{(1)}_\rho$ described in Sec. 2.3.3 (of course $\zeta^r\hat{F} \ll_1 \zeta^r\hat{\mathcal{F}}$ because $|\dot{\zeta}| \leq \ell(\zeta)$), and that $\hat{\mathcal{T}}_n$ fulfills the hypothesis of the preceding lemma. □

In view of (46), there just remains to bound the terms of the series $\sum\hat{\mathcal{R}}_n(\ell(\zeta), i\,\Im\mathrm{m}\,\tau)$ thus defined. As in the proof of Lemma 6, this can be done by considering the series of the Laplace transforms $\tilde{\mathcal{R}}_n(x,\tau)$, which are convergent series of x (when dealing with $\tilde{R}$ and $\tilde{\mathcal{R}}$, we prefer to denote by x the variable which is called z in the case of $\tilde{F}$ or $\tilde{S}$). They are determined inductively by the formulae

$$\tilde{\mathcal{R}}_0 = |\mu|\left(\alpha x\tilde{\mathcal{F}}(x,\tau) - (\alpha x - \rho^{-1})\tilde{\mathcal{F}}_0(x,\tau)\right)$$

$$\tilde{\mathcal{R}}_n = |\mu|\alpha x\sum_{r=1}^{n}\frac{(-1)^r}{r!}\partial_x^r\tilde{\mathcal{F}}(x,\tau)\sum_{n_1+\ldots+n_r=n-r}\tilde{\mathcal{R}}_{n_1}\ldots\tilde{\mathcal{R}}_{n_r}, \qquad n \geq 1.$$

We recognize here the Taylor expansion of an implicit equation: the generating series $\tilde{\mathcal{R}}(x,\tau,\delta) = \sum_{n\geq 0}\tilde{\mathcal{R}}_n(x,\tau)\delta^n$ is solution of

$$\alpha^{-1}x^{-1}\tilde{\mathcal{R}} = |\mu|\left(\tilde{\mathcal{F}}(x-\delta\tilde{\mathcal{R}},\tau) - \left(1-(\alpha\rho x)^{-1}\right)\tilde{\mathcal{F}}_0(x,\tau)\right).$$

Substituting (44) inside but writing $\mathsf{R}(z) = 1 + z\mathsf{S}(z)$, and setting $X = (\rho x)^{-1}$ and $\nu = |\mu|\rho^{-3}\cos\tau$, we find

$$\tilde{\mathcal{R}}(x,\tau,\delta) = \nu(\rho x)^{-1}U\left((\rho x)^{-1}, |\mu|\rho^{-3}\cos\tau, \delta\right),$$

where $U(X,\nu,\delta) = \sum_{n\geq 0}U_n(X,\nu)\delta^n$ solves

$$U = 1 + \alpha\nu(1-\rho\delta\nu X^2U)^{-1}\left(\rho\delta XU + 4f\mathsf{S}(4\nu Xf)\right),$$
$$f = f(X,\nu,\delta,U) = (1-\rho\delta\nu X^2U)^{-1} + X(1-\rho\delta\nu X^2U)^{-2} + X^2(1-\rho\delta\nu X^2U)^{-3}.$$

We conclude by the Implicit Function Theorem: there exist a positive number C and a non-decreasing function Λ (depending only on ρ) such that the function U is holomorphic and bounded by C for $|\delta| \leq 2$ and $|X| \leq \Lambda(|\nu|)^{-1}$. Hence $|x\tilde{\mathcal{R}}_n(x,\tau)| \leq 2^{-n}\rho^{-4}C|\mu\cos\tau|$ for $|x^{-1}| \leq \rho\Lambda(\rho^{-3}|\mu\cos\tau|)^{-1}$, which yields

$$\hat{\mathcal{R}}_n(\zeta,\tau) \leq 2^{-n-1}K|\mu\cos\tau|\, e^{\kappa(\rho^{-3}|\mu\cos\tau|)\zeta}$$

with $K = 2\rho^{-4}C$ and $\kappa = \rho^{-1}\Lambda$.

We deduce the desired result for $\hat{R}(\zeta,\tau)$ on $\mathcal{R}^{(1)} \times (\mathbb{C}/2\pi\mathbb{Z})$ by (46). Changing a little bit the notations, by specializing to $\delta = 1$, we also retain that $\hat{R} \ll_1 \hat{\mathcal{R}}$ with a majorant series $\hat{\mathcal{R}} = \sum_{n\geq 0}\hat{\mathcal{R}}_n(\zeta,\tau)$ which satisfies

$$|\hat{\mathcal{R}}(\zeta,\tau)| \leq K|\mu\cos\tau|\, e^{\kappa(\rho^{-3}|\mu\cos\tau|)\zeta}, \qquad \zeta \in \mathbb{R}^+,\ \tau \in \mathbb{C}. \tag{73}$$

3.2.3 Majorant series for $\hat{S}$

Clearly, the formal series $\hat{S}_n$ of Lemma 13 converge for ζ close to the origin and extend analytically to $\mathcal{R}_\rho^{(1)} \times (\mathbb{C}/2\pi\mathbb{Z})$. We obtain $\hat{S}_n \ll_1 \hat{\mathcal{S}}_n$ by defining

$$\begin{aligned} \hat{\mathcal{S}}_0 &= \hat{\mathcal{R}} \\ \hat{\mathcal{S}}_n &= \sum_{r=1}^{n} \frac{1}{r!} (\zeta^r \hat{\mathcal{R}}) * \sum_{n_1+\ldots+n_r=n-r} \hat{\mathcal{S}}_{n_1} * \ldots * \hat{\mathcal{S}}_{n_r}, \qquad n \geq 1. \end{aligned}$$

The generating series $\tilde{\mathcal{S}}(z,\tau,\delta) = \sum_{n\geq 0} \tilde{\mathcal{S}}_n(z,\tau)\delta^n$ is solution of

$$\tilde{\mathcal{S}} = \tilde{\mathcal{R}}(z - \delta\tilde{\mathcal{S}}, \tau),$$

and we conclude like previously (by enlarging K and κ). Specializing to $\delta = 1$, we retain that $\hat{S} \ll_1 \hat{\mathcal{S}}$ with a majorant series which satisfies the same inequality (73) as $\hat{\mathcal{R}}$.

This ends the proof of Proposition 5.

Notice that the analyticity of $\hat{S}$ can also be obtained by applying the ideas of [Pha89] concerning "resurgent implicit functions".

3.3 Resurgence relations and propagation of analyticity

We are now in a position to begin the proof of Theorem 3 itself. The statement amounts to the analyticity of each $\hat{\phi}_n(\zeta,\tau;\mu)$ in $\mathcal{R} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$ (with at most exponential growth in non-vertical directions on $\mathcal{R}$), with only "simply ramified" singularities which are determined in terms of the formal integral and scalars $f_n^{[\omega]}(\mu)$.

We shall begin with the proof of analyticity in $\mathcal{R}^{(1)} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$, and explain then how to "propagate" it from one sheet of $\mathcal{R}$ to the other by solving linear equations for the alien derivatives.

3.3.1 Analyticity in $\mathcal{R}^{(1)}$ of the $\hat{\phi}_n$'s

Proposition 6 *The formal Borel transforms $\hat{\phi}_n(\zeta,\tau;\mu)$ of the components of the formal integral possess all the property* (A) *of being convergent for ζ close to the origin and defining a holomorphic function on $\mathcal{R}^{(1)} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$ with at most exponential growth in non-vertical directions on $\mathcal{R}^{(1)}$.*

Proof. Property (A) was checked for $\hat{\phi}_0$ in Sec. 2.3 and for $\hat{\phi}_1$ in Sec. 3.2.3 (remember that $\hat{\phi}_1 = \hat{S}$ by definition).

For $n \geq 2$, we use Formula (67), decomposing $\tilde{g}_n$ in the sum of a polynomial part $p_n \in \mathcal{P}[x]$ and a series $\tilde{g}_{[n]} \in x^{-1}\mathcal{P}[[x^{-1}]]$, and then $p_n(z + \tilde{S}(z,\tau),\tau)$ similarly as the sum of $\tilde{P}_n \in \mathcal{P}[z]$ and $\tilde{P}_{[n]} \in z^{-1}\mathcal{P}[[z^{-1}]]$. This is consistent with the notation of (63), and we have

$$\tilde{\phi}_{[n]} = \tilde{P}_{[n]}(z,\tau) + \tilde{g}_{[n]}(z + \tilde{S}(z,\tau),\tau). \tag{74}$$

In view of Proposition 5, Property (A) is immediate for $\hat{P}_{[n]}(\zeta,\tau)$ (expand each monomial $\big(z + \tilde{S}(z,\tau)\big)^j$ and use the stability by convolution of the property to be checked).

For the second term in the right-hand side of (74), we observe that, as $\tilde{g}_n$ verifies (65), it involves the solution $\tilde{g}_{[n]}(x,\tau)$ of the equation

$$(\partial_\tau + \partial_x)\tilde{g}_{[n]} = \tilde{B}_{[n]},$$

where $\tilde{B}_{[n]}$ is obtained from $\tilde{B}_n$ by removing the polynomial part, and that its formal Borel transform can be written

$$\hat{g}_n + \sum_{r\geq 1} \frac{(-1)^r}{r!}(\zeta^r \hat{g}_n) * \hat{S}^{*r}. \tag{75}$$

We first check that the series $\hat{g}_n = \mathcal{B}\tilde{g}_{[n]}$ satisfy Property (A). Indeed,

$$\tilde{B}_2 = x^2(1 + x^{-1}\tilde{R}) \sum_{r\geq 0} (-\partial_x \tilde{R})^r$$

implies that $\hat{B}_2$ satisfies Property (A), since one can use (71) to bound $\big(\zeta \hat{R}(\zeta,\tau)\big)^{*r}$ by $K|\mu| \cosh(\Im\mathrm{m}\,\tau) \frac{\ell(\zeta)^{2r-1}}{(2r-1)!} e^{\kappa \ell(\zeta)}$, thus $\hat{g}_2 = -\mathcal{E}_0 \hat{B}_2$ satisfies it too, and the same is true for the next functions $\hat{B}_n$ and $\hat{g}_n = -\mathcal{E}_0 \hat{B}_n$ by induction on n.

We then see that (75) is a convergent series of holomorphic functions for each $n \geq 2$ (use (71) to bound $\hat{S}^{*r}$). □

Corollary 3 *The singularities $\overset{\triangledown}{\varphi}_n$ defined by (68) for $n \geq 0$ belong to the spaces $\mathrm{RES}^{(1)}_{\theta,\beta}$ for any $\theta \in \frac{\pi}{2} + \pi\mathbb{Z}$ and $\beta \leq \pi$.*

Proof. This was already noticed in the case of $\overset{\triangledown}{\varphi}_0 = {}^b\hat{\varphi}_0$ in Sec. 2.4.2, as a consequence of Theorem 1. Proposition 6 shows in the same way that the remaining singularities $\overset{\triangledown}{\varphi}_n$ also satisfy the property described in Definition 6. □

3.3.2 Proof of the Bridge equation for $\omega = \pm i$

For the time being, we have at our disposal only two alien derivations that can be tested on the singularities $\overset{\triangledown}{\varphi}_n$, namely Δ_i and Δ_{-i}.[7] According to Theorem 2, especially after the explanations of Sec. 2.4.3, we already know that $\Delta_i \overset{\triangledown}{\varphi}_0 = f_0^{[i]}(\mu) e^{i\tau} \exp_*\big(-{}^b(i\hat{S})\big)$, and there is of course a similar relation for $\Delta_{-i}\overset{\triangledown}{\varphi}_0$; these first relations can be written in the formal model:

$$\dot{\Delta}_i \tilde{\varphi}_0 = f_0^{[i]}(\mu) e^{-i\tilde{\varphi}_1}, \quad \dot{\Delta}_{-i} \tilde{\varphi}_0 = f_0^{[-i]}(\mu) e^{i\tilde{\varphi}_1}.$$

We can now easily derive analogous relations for the remaining singularities $\overset{\triangledown}{\varphi}_n$, by following exactly the same chain of reasoning as in Sec. 2.4.3.

[7]In fact, we should speak of all the alien derivations $\Delta_{e^{i\theta}}$ with $\theta \in \frac{\pi}{2} + \pi\mathbb{Z}$, but we know in advance that our singularities (since they have a regular minor) belong to an algebra where only two alien derivations among them are to be distinguished.

Notice that, due to the antisymmetry of $\tilde{\phi}_0$, $f_0^{[i]} = f_0^{[-i]}$ and, by antisymmetry of $\tilde{\phi}_1$, $\Delta_{-i}\tilde{\phi}_0 = -\Delta_i\tilde{\phi}_0$.

It is more efficient to deal with the generating series $\overset{\triangledown}{\phi} = \sum c^n \overset{\triangledown}{\phi}_n$: let $\overset{\triangledown}{X} = \Delta_\omega \overset{\triangledown}{\phi}$, i.e.

$$\overset{\triangledown}{X} = \sum_{n\geq 0} c^n \overset{\triangledown}{X}_n, \qquad \overset{\triangledown}{X}_n = \Delta_\omega \overset{\triangledown}{\phi}_n \in \mathrm{SING}_{\theta,\beta},$$

for $\omega = e^{i\theta}$, with $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ and $\beta = \pi$. Observe that, at this stage, the nature of the singularity of each $\hat{\phi}_n$ still being unknown, there is no obvious formal counterpart for the singularities $\overset{\triangledown}{X}_n$ (for $n \geq 1$, that is). But the rules of alien calculus show that these singularities satisfy linear equations obtained by applying Δ_ω to (57) and (58); the corresponding equation for the generating series is

$$\partial_\tau \overset{\triangledown}{X} + \overset{\triangledown}{D} * \partial \overset{\triangledown}{X} = \omega \overset{\triangledown}{D} * \overset{\triangledown}{X}, \qquad \overset{\triangledown}{D} = -\frac{1}{4}\delta^{(2)} * \partial \overset{\triangledown}{\phi}. \tag{76}$$

On the other hand, we can easily produce a series of singularities $\overset{\triangledown}{Z} = \sum_{n\geq 0} c^n \overset{\triangledown}{Z}_n$ which satisfies

$$\partial_\tau \overset{\triangledown}{Z} + \overset{\triangledown}{D} * \partial \overset{\triangledown}{Z} = -\omega \overset{\triangledown}{D} * \overset{\triangledown}{Z}. \tag{77}$$

Consider indeed $\overset{\triangledown}{Z} = \exp_*(-\omega z + \omega \partial_c \overset{\triangledown}{\phi})$ (which must be understood, using the expansion of the exponential, as a formal series $\sum c^n \overset{\triangledown}{Z}_n$ with coefficients in $\mathrm{SING}_{\theta,\beta}$: $\overset{\triangledown}{Z}_0 = e^{-\omega\tau + \omega^b \hat{S}}$, etc.). In fact, Eq. (77) is a consequence of the equation

$$(\partial_\tau + \tilde{D}\partial_z)\partial_c \phi = 0$$

which is simply the linearization of Eq. (1) around the whole formal integral $\tilde{\phi}$.

Combining Eq. (76) and(77), we obtain

$$\partial_\tau \overset{\triangledown}{\varphi} + \overset{\triangledown}{D} * \partial \overset{\triangledown}{\varphi} = 0, \qquad \overset{\triangledown}{\varphi} = \overset{\triangledown}{X} * \overset{\triangledown}{Z},$$

which is solved by expanding in powers of c and reasoning as at the end of Sec. 2.4.3: we find indeed $\partial_\tau \overset{\triangledown}{\varphi}_0 + \overset{\triangledown}{D}_0 * \partial \overset{\triangledown}{\varphi}_0 = 0$, which is Eq. (54), thus $\overset{\triangledown}{\varphi}_0$ must be proportional to δ ($\overset{\triangledown}{\varphi}_0 = f_0^{[\omega]}(\mu)\delta$), $\partial \overset{\triangledown}{\varphi}_0$ must vanish, and we obtain by induction that every $\overset{\triangledown}{\varphi}_n$ is proportional to δ

The upshot is a sequence of proportionality factors $\left(f_n^{[\omega]}(\mu)\right)$ such that

$$\Delta_\omega \overset{\triangledown}{\phi} = \left(\sum_{n\geq 0} c^n f_n^{[\omega]}(\mu)\right) \exp_*(\omega z - \omega \partial_c \overset{\triangledown}{\phi}), \qquad \omega = \pm i. \tag{78}$$

And now, expanding with respect to c, we see that each $\overset{\triangledown}{\phi}_n$ admits a formal Laplace transform in $\mathcal{P}[z][[z^{-1}]]$; the relations corresponding to (78) and its expansion in the formal model are precisely the resurgent relations indicated in Theorem 3 in the case of $\omega = \pm i$.

3.3.3 Alien derivations as a tool to explore the Riemann surface $\mathcal{R}$

Let $\mathrm{RES}^{(1)}$ denote the algebra consisting of all singularities which belong to $\mathrm{RES}^{(1)}_{\theta,\pi}$ for $\theta \in \frac{\pi}{2} + \pi\mathbb{Z}$ and admit a regular minor which extends analytically to $\mathcal{R}^{(1)}$. We know by Corollary 3 that each $\overset{\triangledown}{\phi}_n$ belongs to $\mathrm{RES}^{(1)}$.

When expanded with respect to c, Eq. (78) shows that the singularities $\overset{\triangledown}{\phi}_n$ belong in fact to a subspace that we can denote by $\mathrm{RES}^{(2)}$. This is the subspace of $\mathrm{RES}^{(1)}$ consisting of the members $\overset{\triangledown}{\varphi}$ of $\mathrm{RES}^{(1)}$ whose alien derivatives $\Delta_{\pm i}\overset{\triangledown}{\varphi}$ belong to $\mathrm{RES}^{(1)}$ and whose minors $\hat{\varphi}$ extend analytically along the paths which issue from the origin and cross the imaginary axis exactly once and between $2i$ and $3i$ or between $-2i$ and $-3i$.

The arguments (and the notations $\mathrm{RES}^{(1)}$, $\mathrm{RES}^{(2)}$) are essentially the same as in [GS01, p. 588]. The idea is that the alien derivations provide a tool to explore the Riemann surface $\mathcal{R}$, since, when $\overset{\triangledown}{\psi}_\pm = \Delta_{\pm i}\overset{\triangledown}{\varphi}$, the determination of $\hat{\varphi}$ in any of the four half-sheets accessed by crossing the imaginary axis between i and $2i$ or between $-i$ and $-2i$ can be expressed in terms of the principal determinations of $\hat{\varphi}$, $\hat{\psi}_+$ and $\hat{\psi}_-$ (this yields expressions like $\hat{\varphi}(\overset{\bullet}{\zeta}) \pm \hat{\psi}_+(\overset{\bullet}{\zeta} - i)$ or $\hat{\varphi}(\overset{\bullet}{\zeta}) \pm \hat{\psi}_-(\overset{\bullet}{\zeta} + i)$).

The process can be continued because $\mathrm{RES}^{(2)}$ is a subalgebra on which not only the first alien derivations $\Delta_{\pm i}$ are defined, but also the operators $\Delta_{\pm i} \circ \Delta_{\pm i}$ and $\Delta_{\pm 2i}$. The operators Δ_{2i} and Δ_{-2i} are defined by (69) (which is meaningful when $\omega = \pm 2i$ and $\overset{\triangledown}{\varphi} \in \mathrm{RES}^{(2)}$) and they satisfy the Leibnitz rule.

This allows us to "alien differentiate" Eq. (78) (applying $\Delta_{\pm i}$ after having expanded it) and to repeat the arguments of Sec. 3.3.2 with $\Delta_{\pm 2i}$, obtaining expressions of all the (so far) computable alien derivatives in terms of singularities which are all known to belong to $\mathrm{RES}^{(2)}$. Again, these formulas can be interpreted as a piece of information concerning the determinations of the minors $\hat{\phi}_n$ on farther sheets of $\mathcal{R}$, sufficient to establish that the singularities $\overset{\triangledown}{\phi}_n$ belong to $\mathrm{RES}^{(3)}$, etc.

The reader is referred to [CNP93b], p. 210–216, from which we have borrowed the title of the present section. This ends the proof of Theorem 3.

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